: Conservation Laws

We consider the IVP for the scalar conservation law

$$(1)\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u(x,0) = g(x) & \text{in } \mathbb{R} \end{cases}$$

Recall, after writing the equation as

$$u_t + F'(u)u_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$

Characteristics are

$$\dot{x} = F'(u(x(t),t)) = F'(g(x_0))$$

since $\dot{z} = \frac{d}{dt}u(x(t),t) = 0$.

In general the characteristics will cross leading to multiple possibly singular solutions ~~not all of which are not smooth~~.

• Weak solutions

Suppose to start that $\begin{cases} u_t + F(u)_x = 0 \\ u(x,0) = g(x) \end{cases}$,

u is a smooth solution.

Let $v: \mathbb{R} \times [0,\infty) \to \mathbb{R}$ smooth

w/ cpt support

we call v a <u>test function</u> (like when

we introduced distributions)

multiply the PDE by v and integrate by parts

$$0 = \int_0^\infty \int_{\mathbb{R}} (u_t + F(u)_x) v \, dx \, dt$$

$$\text{IBP} = -\int_0^\infty \int_{\mathbb{R}} u v_t + F(u) v_x \, dx dt - \int_{\mathbb{R}} u(x,0) v(x,0) \, dx$$

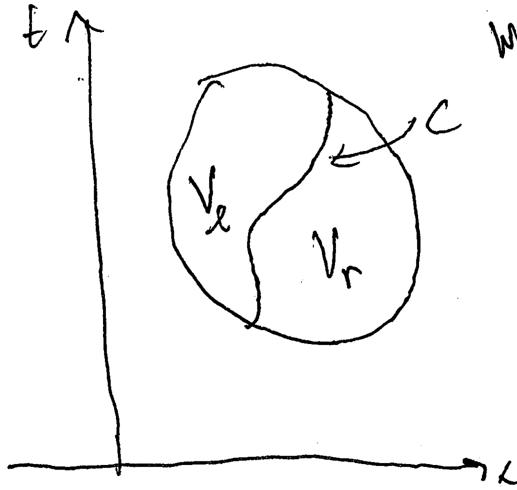
using $u(x,0) = g(x)$ we get

(2) $$\boxed{\int_0^\infty \int_{\mathbb{R}} u v_t + F(u) v_x \, dx dt + \int_{\mathbb{R}} g v\big|_{t=0} \, dx = 0 \quad \text{for all test functions } v}$$

<u>Def</u>: We say $u \in L^\infty(\mathbb{R} \times (0,\infty))$ is an <u>integral solution</u> of (1) if (2) holds for every test functions

: what kind of non-smooth solutions can we have?

We consider a very particular kind of discontinuous u



We suppose that u is an integral solution of (1) in $V \subseteq \mathbb{R}\times(0,\infty)$ with u smooth ~~in V~~ on either side of a smooth curve C parametrized by ~~[illegible]~~. $(\gamma(t), t)$

V_e is the part of V to the left of C

V_r " " " " " to the right of C

And we suppose u smooth in V_e and V_r, and u, u_t, u_x unif cts in V_e, V_r.

Choosing v w/ cpct support in V_e

$$0 = \int_0^\infty \int_{-\infty}^\infty u v_t + F(u) v_x \, dx\, dt = -\int_0^\infty \int_{-\infty}^\infty (u_t + F(u)_x) v \, dx\, dt$$

since u smooth in V_e and v cpctly supported in V_e.

$\Rightarrow$ $u_t + F(u)_x = 0$ ptwise in V_e

same result in V_r

now let v w/ cpct support in V, not necessarily vanishing on C. Then

$$0=\int_0^\infty\int_{\mathbb{R}} uv_t + F(u)v_x\,dx\,dt$$

$$=\iint_{V_\ell} uv_t+F(u)v_x\,dx\,dt + \iint_{V_r} uv_t + F(u)v_x\,dx\,dt$$

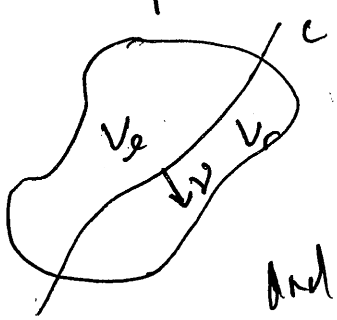
since v cptly supported in V

$$\iint_{V_\ell} uv_t+F(u)v_x = -\iint_{V_\ell}[u_t+F(u)_x]\,v\,dx\,dt$$

$$+\int_C [u_\ell \nu_t + F(u_\ell)\nu_x]\,v\,ds$$

$$= \int_C v\left(u_\ell\nu_t + F(u_\ell)\nu_x\right)ds$$

where ν is unit normal to C pointing from V_ℓ into V_R, $\nu=(\nu_x,\nu_t)$



~~and~~ $\cancel{u_\ell(x) = \lim_{\substack{y\to x\\ y\in V_\ell}}}$

and $u_\ell(x,t)$ is the "left limit"

$$u_\ell(x,t) = \lim_{V_\ell\ni(y,s)\to(x,t)} u(y,s)$$

Similarly

$$\iint_{V_r} u v_t + F(u) v_x \, dx dt = -\int_C [u_r v_t + F(u_r) v_x] v \, ds$$

so

$$\int_C [(u_\ell - u_r) v_t + (F(u_\ell) - F(u_r)) v_x] v \, ds = 0$$

for every v a smooth cpct support in V.

$$\Rightarrow \quad (u_\ell - u_r) v_t + (F(u_\ell) - F(u_r)) v_x = 0$$

pointwise on C.

now using the parametrization

$$C = \{ (\gamma(t), t) : t \in I \}$$

unit tangent vector is $\dfrac{(\dot\gamma, 1)}{\sqrt{1 + |\dot\gamma|^2}}$

unit normal is $\dfrac{(-1, \dot\gamma)}{\sqrt{1 + |\dot\gamma|^2}}$

~~Quasi-linear equations~~

~~(linear in p)~~

~~also just need equations for $\mathbb{X}, Z$~~

~~not p.~~

~~Fully Nonlinear (nonlinear in p)~~

~~now need full method of characteristics.~~

~~example~~

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so $\quad F(u_l) - F(u_r) = \dot{\gamma}(t)(u_l - u_r)$

or $\quad \dot{\gamma} = \dfrac{F(u_l) - F(u_r)}{u_l - u_r}$

we call $\quad [u] = u_l - u_r$ = jump of u across curve C

$[F(u)] = F(u_l) - F(u_r)$ = jump of $F(u)$

$\sigma = \dot{\gamma}$ = speed of curve C

then $\quad \sigma = \dfrac{[F(u)]}{[u]}$ Rankine-Hugoniot Condition

Example 1 : Shock wave

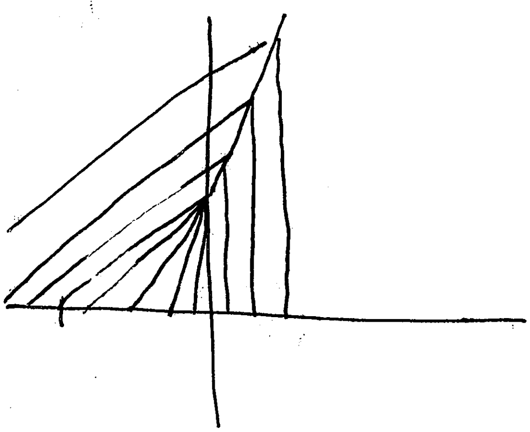
$$(1)\ \begin{cases} \cancel{u_t + \frac{(u_x^2)}{2}} \quad u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u = g & \text{on } \mathbb{R}\times\{t=0\} \end{cases}$$

w/

$$g(x) = \begin{cases} 1 & x \le -1 \\ -x & -1 < x < 0 \\ 0 & x > 0 \end{cases}$$

We already solved for $t \le 1$

$$u(x,t) = \begin{cases} 1 & x \le t-1 \\ -\frac{x}{1-t} & t-1 < x < 0 \\ 0 & x > 0 \end{cases}$$



for $t \ge 1$ let's define a shock solution

$$u(x,t) = \begin{cases} 1 & x \le s(t) \\ 0 & x > s(t) \end{cases} \quad \text{for } t > 1$$

w/ $s(1) = 0$

to be a weak solution we need

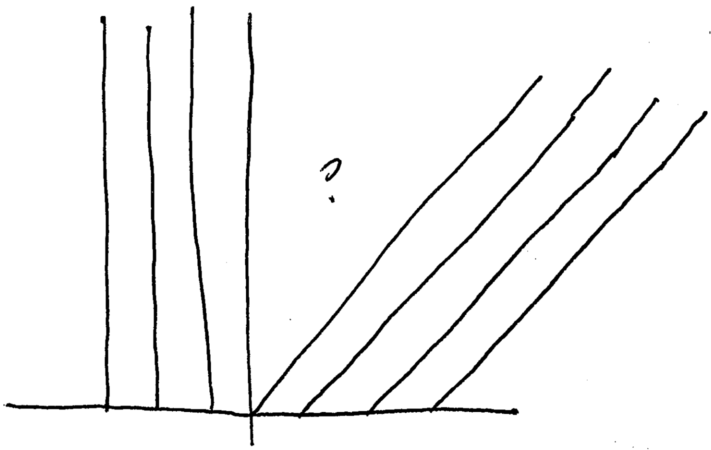
$$\dot{s} = \frac{F(u_l) - F(u_r)}{u_l - u_r} = \frac{u_l^2 - u_r^2}{2(u_l - u_r)} = \frac{1}{2}$$

so

$$\boxed{s(t) = s(1) + \frac{1}{2}t = 0 + \frac{1}{2}t}$$

Example 2 Consider a similar problem

$$g(x) = \begin{cases} 0 & x<0 \\ 1 & x \geq 0 \end{cases}$$



don't know what happens in $0<x<t$ wedge.

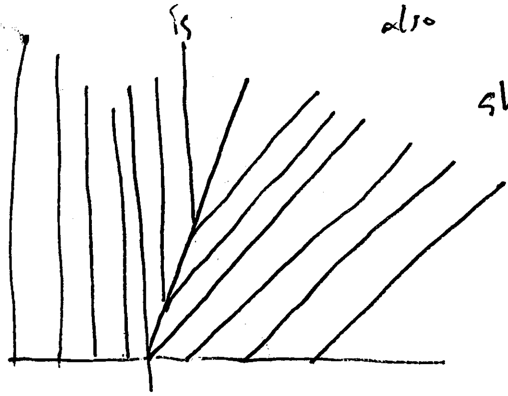
e.g. $$u_1(x,t) = \begin{cases} 0 & x< \frac{t}{2} \\ 1 & x > \frac{t}{2} \end{cases}$$

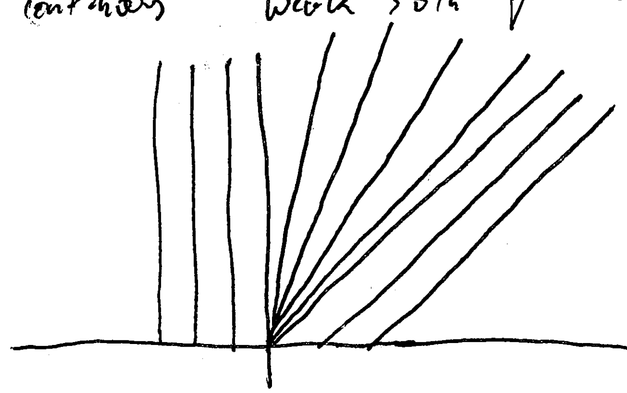
Rankine-Hugoniot: $\frac{u_l^2-u_r^2}{2(u_l-u_r)} = \frac{0-1}{2(0-1)} = \frac{1}{2} = \dot{s}$ ✓

So u_1 is a weak solution.

$$u_2(x,t) = \begin{cases} 0 & x<0 \\ x/t & 0\leq x\leq t \\ 1 & x \geq t \end{cases}$$ rarefaction wave

is also a continuous weak sol^n of (1).





Lax Entropy Condition

Clearly we need an additional condition in order to specify a weak solution:

entropy condition

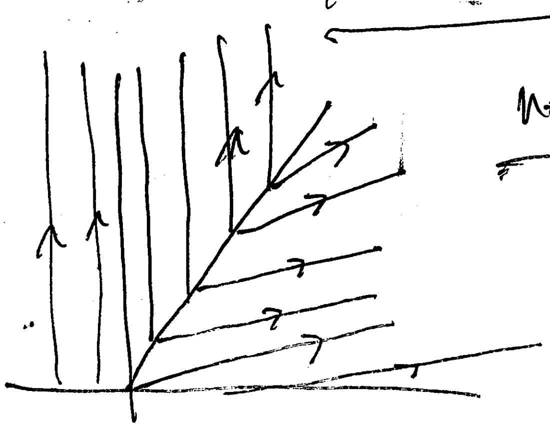
(1) no discontinuities along backwards characteristics

$(x + F'(u(x,t))s, s)$ for $s \leq t$,

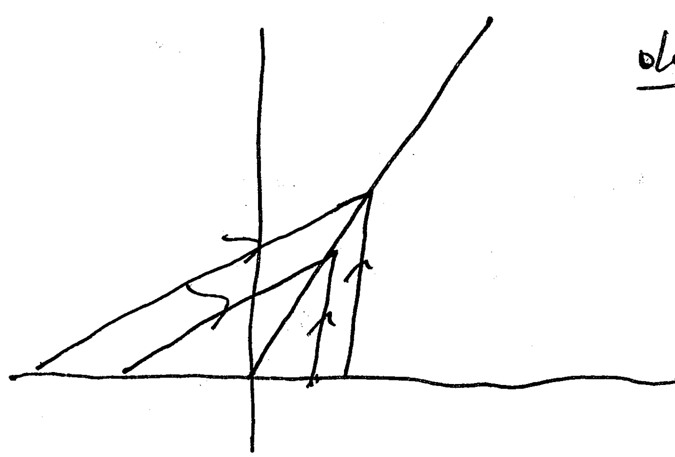
doesn't encounter any shocks

(2) characteristics go into a shock

$$\boxed{F'(u_l) > \sigma > F'(u_r)}$$



not ok



ok

when F is uniformly convex ($F'' \geq \theta > 0$)

$\Rightarrow$ F' strictly increasing

entropy condition $\iff$ $u_\ell > u_r$

shocks must jump down from left to right

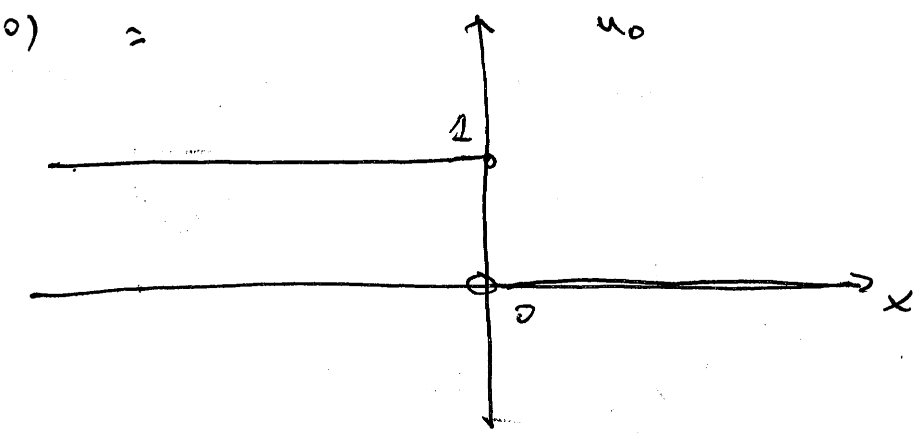
Note: ~~sats~~

weak solution + entropy condition $\Rightarrow$ uniqueness

but there will always be

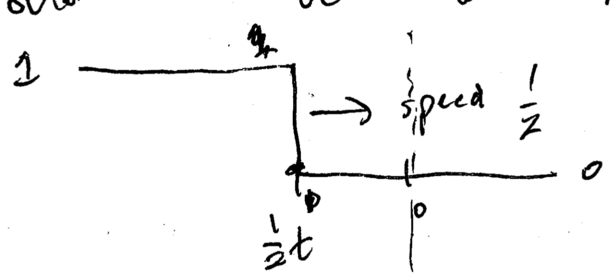
backwards non-uniqueness

e.g. $u(x,0) =$ u_0

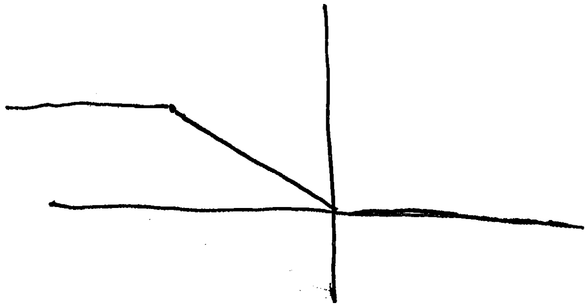


where did this state come from?

could be a shock



could be rarefaction



$t = -1$

(this has backwards uniqueness)

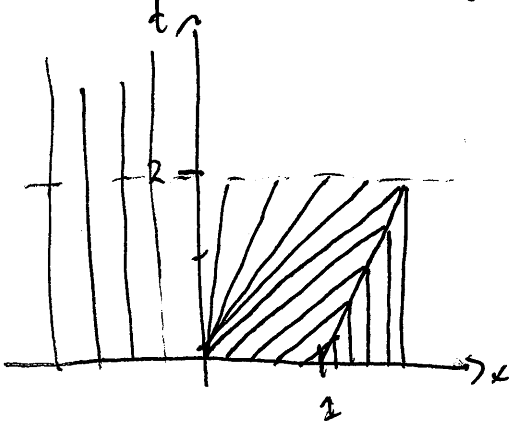
family of backwards solutions

information is lost when a shock forms

(entropy increases as t increases)

Example (piecewise smooth data)

$$\begin{cases} u_t + u u_x = 0 & t > 0 \\ u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases} & t = 0 \end{cases}$$



characteristics

(1) shock $\gamma(t)$ started at

$$\gamma(0) = 1$$

w/ $\dot\gamma(t) = \frac{u_\ell + u_r}{2} = \frac{1+0}{2} = \frac{1}{2}$

at least until the characteristic

from $0, x=t$, catches up w/ $\gamma = 1 + \frac{t}{2}$

at $t_0 = 2$

(2) rarefaction in $0 < x < t$, $t \le 2$

$$u(x,t) = \frac{x}{t}$$

$t \geq 2$
$$u(x,t) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < \gamma(t) \\ 0 & x > \gamma(t) \end{cases}$$

what is $\gamma(t)$? $\gamma(2) = 1 + \frac{2}{2} = 2$

$$\dot\gamma = \frac{u_\ell(\gamma(t),t) + u_r(\gamma(t),t)}{2} \qquad \text{(Rankine-Hugoniot)}$$

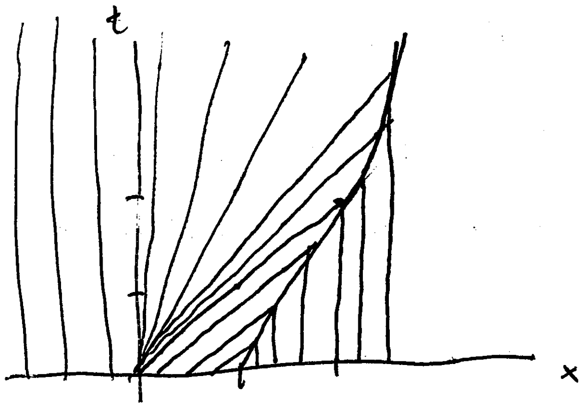
$$= \frac{\gamma(t)/t + 0}{2} = \frac{\gamma(t)}{2t}$$

ODE for γ.

$$\begin{cases} \dot{\gamma} = \dfrac{\gamma}{2t} & t > 2 \\ \gamma(2) = 2 \end{cases} \Rightarrow \quad \gamma(t) = \sqrt{2}\, t^{1/2}$$

So for $t \geq 2$

$$u(x,t) = \begin{cases} 0 & x \leq 0 \\ x/t & 0 < x < \sqrt{2}\, t^{1/2} \\ 0 & x > \sqrt{2}\, t^{1/2} \end{cases}$$



Similar situations make up solution for more general smooth initial data.

TO

The Riemann Problem (general scalar conservation law)

(RP) $\begin{cases} u_t + (f(u))_x = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u(x,0) = \begin{cases} u_L & \text{if } x<0 \\ u_R & \text{if } x>0 \end{cases} \end{cases}$

if we can figure out how to solve all the Riemann problems we can build up solution w/ general piecewise smooth initial data.

The (RP) has a scaling invariance

for any $a>0$

$$\bar{x} = ax \qquad \bar{t} = at$$

$\cancel{\bar{u}(\bar{x},\bar{t}) = u(ax,at)}$ $v(x,t;a) = \cancel{u(x,t)}\ u(ax,at)$ solves

$v(x,0;a) = u(x,0)$ since initial data is scale invariant

and $\partial_t v = a\,\partial_t u(ax,at) = -af'(u(ax,at))\,u_x(ax,at)$

$$= -(f(v))_x$$

so $v_t + (f(v))_x = 0 \implies v$ solves (RP) as well.

Since we want uniqueness for (RP) we need for the solution to satisfy

$$v(x,t;a) = u(x,t) \quad \text{for all } a>0.$$

i.e. $u(x,t) = u(ax, at)$ for all $a>0$, $x \in \mathbb{R}$, $t>0$.

$\Rightarrow$ (taking $a = t$) that

$$u(x,t) = u(\tfrac{x}{t}, 1) = \bar{u}(\tfrac{x}{t})$$

i.e. u is constant along rays

$x = ct$ through the origin in the x,t plane.

When $\bar{u}$ is cts this is called a

<u>centered rarefaction wave</u>

~~Calling~~

Calling $\xi = \frac{x}{t}$

$u(x,t) = \bar{u}(\xi)$ solves ~~burgers~~ conservation law

$$\Rightarrow 0 = u_t + f(u)_x = \bar{u}' \frac{\partial \xi}{\partial t} + f'(\bar{u})\,\bar{u}' \frac{\partial \xi}{\partial x}$$

$$= -\frac{\xi}{t}\bar{u}' + f'(\bar{u})\,\bar{u}' \frac{1}{t} \quad \text{in } \mathbb{R}\setminus\{0\}$$

so $\bar{u}'(f'(\bar{u}) - \xi) = 0$ implies

that either $\bar{u}'(\xi)$ or $\underbrace{f'(\bar{u}) = \xi}$ for every $\xi \in \mathbb{R}\setminus\{0\}$

this equation will generate a rarefaction wave in any interval of $\mathbb{R}$ where

$f'' \neq 0$ i.e. f' either increasing or decreasing

i.e. f either convex or concave.

example $\boxed{\text{traffic flow model}}$

$\rho(x,t)$ = density of cars at $x \in \mathbb{R}$, $t \in \mathbb{R}$.

$v(\rho) = v_m(1 - \rho/\rho_m)$ — velocity of traffic

v_m = maximum velocity in no traffic

only physical for $0 \le \rho \le \rho_m$

flux $\quad f(\rho) = \rho v(\rho) = v_m \rho (1 - \rho/\rho_m)$

$$\begin{cases} \rho_t + f(\rho)_x = 0 & \mathbb{R} \times \{t > 0\} \\ \rho(x,0) = \rho_0(x) & t = 0 \end{cases}$$

flux is concave $\Rightarrow$ f' decreasing

$f'(\rho_\ell) > f'(\rho_r) \Rightarrow \rho_\ell < \rho_r$

Shocks occur only for upward jumps.

rarefaction $\quad f'(\bar\rho) = \xi$

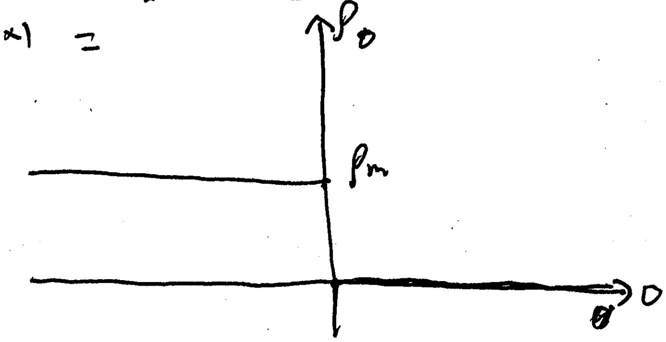
$$v_m - \frac{2 v_m \bar\rho}{\rho_m} = \xi \qquad \text{so}$$

~~$\bar\rho(\xi) = \frac{\rho_m \xi}{2 v_m}$~~ $\longrightarrow$

$$\bar{\rho}(\xi) = \frac{\rho_m}{2} - \frac{\rho_m}{2v_m}\xi = \frac{\rho_m}{2}\left(1 - \frac{\xi}{v_m}\right)$$

<u>red light turns green</u>

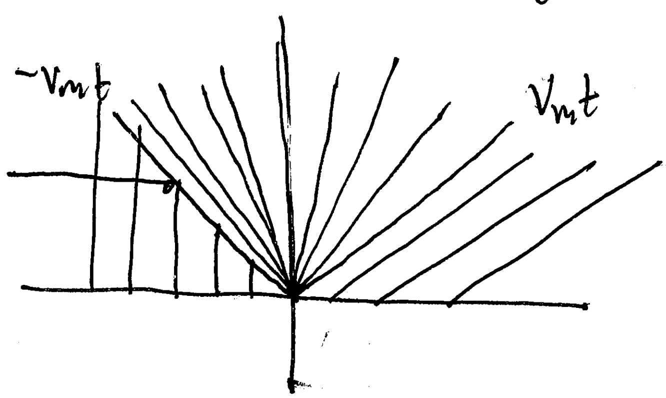
$\rho_0(x) =$

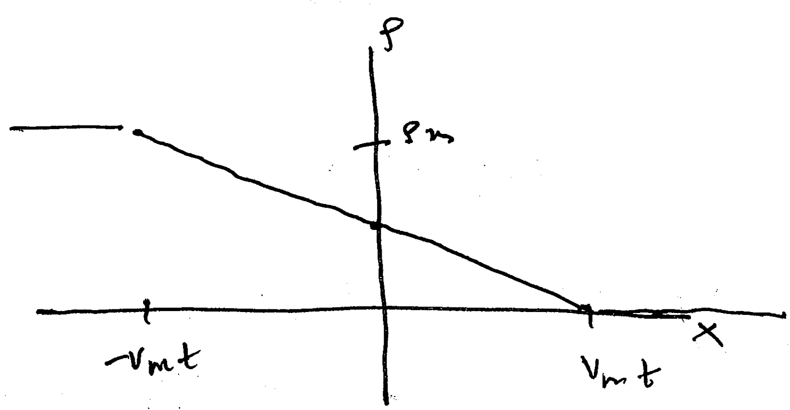


(red light turns green at $t=0$)

$\emptyset \to$ rarefaction connecting ρ_m to 0

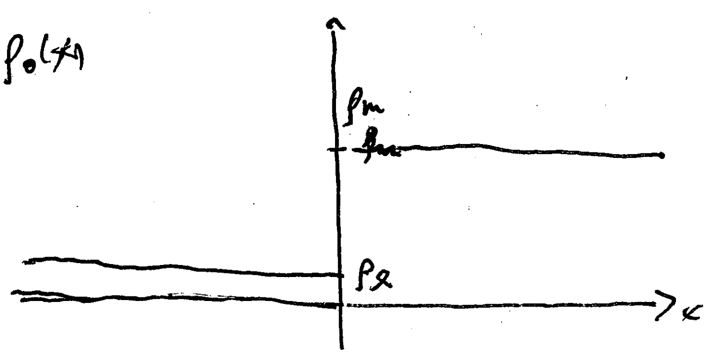
$$\rho(x,t) = \begin{cases} \rho_m & \frac{x}{t} \le -v_m \\ \frac{\rho_m}{2}\left(1 - \frac{x}{v_m t}\right) & -v_m t < x < v_m t \\ 0 & x \geq v_m t \,. \end{cases}$$





<u>traffic jam ahead:</u>

$\rho_0(x)$



shock w/ speed

$$\sigma = \frac{[f]}{[\rho]} = \frac{f(\rho_\ell) - f(\rho_m)}{\rho_\ell - \rho_m}$$

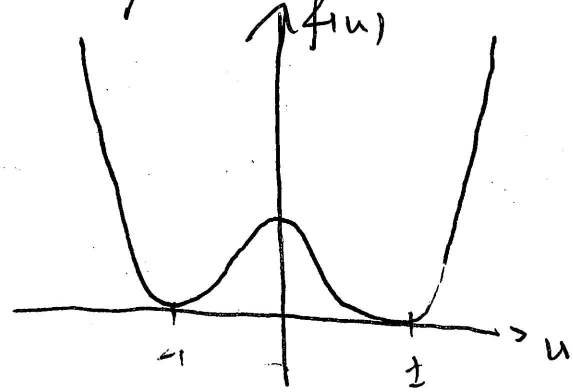
$$f(\rho_\ell) - f(\rho_m) = f(\rho_\ell) - 0 = V_m \rho_\ell \left(1 - \frac{\rho_\ell}{\rho_m}\right) = \frac{V_m \rho_\ell}{\rho_m}(\rho_m - \rho_\ell)$$

$\cancel{f}$ $\sigma = \cancel{\ } - \frac{V_m \rho_\ell}{\rho_m}$ < 0 so shock

propagates ~~backwards~~ left w/ speed $\sigma = -\frac{V_m \rho_\ell}{\rho_m}$

Nonconvex flux

e.g.



or $\left[u_t + (u^3)_x = 0 \right.$

rarefaction: $f'(\bar{u}) = 3\bar{u}^2 = \xi$

so $\bar{u}(\xi) = \pm\left(\frac{\xi}{3}\right)^{1/2}$ for $\xi > 0$.

so rarefactions can join u_ℓ and u_r if

$u_r < u_\ell \leq 0$ or $0 \leq u_\ell < u_r$.

(speed $f'(\bar{u})$ needs to increase left to right)