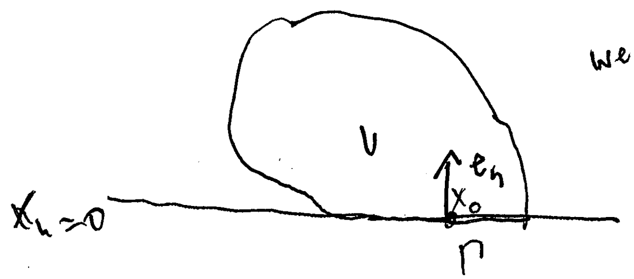
Non-characteristic Boundary data

We are given $x_0 \in \Gamma$ wish to solve $F(Du, u, x) = 0$ in U (near x_0)



we may as well assume that $\nu(x_0) = e_n$, and that Γ is flat near x_0 lying in $\{x_n = 0\}$.

This can be achieved for smooth curved bdry by a change of variables.

Now we wish to solve the characteristic ODE with $X(0)$ near x_0, but we need initial data for p, z.

$$p(0) = p_0 \quad , z(0) = z_0 \quad , X(0) = x_0$$

clearly $\boxed{z_0 = g(x_0)}$ from boundary condition of u.

since $u(y) = g(y)$ for $y \in \Gamma$

$$\boxed{\partial_{x_i} u(x_0) = \partial_{x_i} g(x_0) \quad \text{for} \quad i = 1, \dots, n-1 .}$$

Furthermore we require the PDE to hold

$$\boxed{F(p_0, z_0, x_0) = 0}$$

this gives n-eqns for the n unknowns p_0^i, but there may be no solution or multiple solutions for p_0^n.

Not only must we solve the compatibility conditions at x_0, but also near x_0,

we need a mapping $q(y)$ so that

$$\text{C1} \quad \begin{cases} q^i(y) = g_{x_i}(y) \\ F(q(y), g(y), y) = 0 \end{cases} \quad \text{for } y \text{ near } x_0$$

w/ $q(y) = p_0$.

Lemma There exists a unique solution $q(\cdot)$ of (1) for all $y \in \Gamma$ sufficiently near x_0 as long as

$$F_{p_n}(p_0, z_0, x_0) \neq 0.$$

The proof uses the Implicit fcn theorem.

proof: define $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$$G^i(p,y) = p_i - g_{x_i}(y) \qquad i = 1, \dots, n-1$$

$$G^n(p,y) = F(p, g(y), y)$$

$G(p_0, x_0) = 0$ by assumption and

$$\underset{\left(\frac{\partial G^i}{\partial p_j}\right)_{ij}}{D_p G(p_0, x_0)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & & & 1 & 0 \\ F_{p_1}(p_0, z_0, x_0) & \cdots & & & F_{p_n}(p_0, z_0, x_0) \end{pmatrix}$$

$$\det D_p G(p_0, x_0) = F_{p_n}(p_0, z_0, x_0) \neq 0$$

so $D_p G(p_0, x_0)$ non-degenerate

implicit fcn thm $\Rightarrow$ $\exists$ a unique solution $q(y)$

$F(q(y), g(y), y) = 0$ for y in a small nbhd of x_0.

~~Solving~~

~~Branching~~

$\cancel{x = \Phi(y,s)}$ ~~ys~~

Now that we have initial data

$(q(y), g(y), y)$ for the characteristic ODEs

for y close to x_0

for any such $y = (y_1, \dots, y_{n-1}, 0)$ we solve

$$\begin{cases} \dot{p} = -D_x F(p,z,\Phi) - p\, D_z F(p,z,\Phi) \\ \dot{z} = p \cdot D_p F(p,z,\Phi) \\ \dot{\Phi} = D_p F(p,z,\Phi) \end{cases}$$

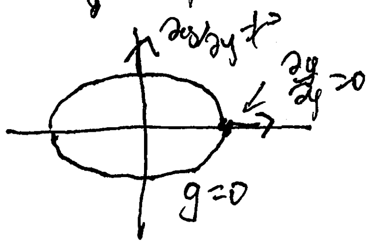
w/ $\cancel{\Phi(y,} \cancel{\Phi(0,y) = q(y)}$ initial data

$$\left[\begin{array}{l} p(0,y) = q(y) \\ z(0,y) = g(y) \\ \Phi(0,y) = y \end{array}\right.$$

Implicit function thm :

e.g. $\quad f: \mathbb{R}^2 \to \mathbb{R}$

$f(x_0, y_0) = 0$



0-level set is locally a C^1 surface near (x_0, y_0) if $\nabla f(x_0, y_0) \neq 0$

if $\nabla f(x_0,y_0)\cdot(0,1) = \frac{\partial f}{\partial y}(x_0, y_0) \neq 0$

then $\{f=0\}$ is locally a graph

$\exists$ g ~~s.t.~~ C^1 s.t.

$f(x, g(x)) = 0$ for x near x_0

$g(x_0) = y_0$.

$x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $U \subseteq \mathbb{R}^{n+m}$ $\quad$ (x,y) pts in $\mathbb{R}^{n+m}$

$f: U \to \mathbb{R}^m$ is C^1 , $f = (f_1, \ldots, f_m)$

$$D_y f = \begin{pmatrix} f^1_{y_1} & \cdots & f^1_{y_m} \\ \vdots & & \vdots \\ f^m_{y_1} & \cdots & f^m_{y_m} \end{pmatrix} \quad m \times m \text{ matrix}$$

Thm Assume $f \in C^1(U; \mathbb{R}^m)$ and $|\det D_y f(x_0, y_0)| \neq 0$.

Then $\exists$ V open $\subset U$ w/ $(x_0, y_0) \in V$, $W \subseteq \mathbb{R}^n$ w/ $x_0 \in W$ and a C^1 mapping $g: W \to \mathbb{R}^m$ s.t.

(i) $g(x_0) = y_0$

(ii) $f(x, g(x)) = z_0 \left(= f(x_0, y_0)\right)$

and

(iii) if $(x,y) \in V$ w/ $f(x,y) = z_0$ then $y = g(x)$

(iv) if $f \in C^k$ then $g \in C^k$.

Lem Assume $F_{p_n}(q_0, z_0, x_0) \neq 0$. Then $\exists$ an open interval $I \subseteq \mathbb{R}$ containing 0, a nbhd W of x^0 in $\Gamma \subseteq \mathbb{R}^{n-1}$, and a nbhd V of x_0 in $\mathbb{R}^n$ such that for each $x \in V$ there exist unique $s \in I$, $y \in W$ such that

$$x = X(s, y)$$

and $x \mapsto s, y$ is C^2 mapping.

Proof: ~~$X(x_0)$~~ $X(0, x_0) = x_0$ so we can use inverse fun thm to locally invert $X : \mathbb{R}^n \to \mathbb{R}^n$ as long as

$$\det DX(0, x_0) \neq 0.$$

Since $X(0, y) = y$

$$\cancel{\frac{\partial}{\partial y_j} X(0, x_0^0) =}$$

$$\frac{\partial}{\partial y_i} X^j(0, x_0^0) = \begin{cases} \delta_{ij} & j \neq n \\ 0 & j = n \end{cases}$$

and since

$$\frac{\partial X^i}{\partial s}(0, x_0) = F_{p_i}(p_0, z_0, x_0) \quad \text{from the ODE}$$

$$DX(0, x_0) = \begin{pmatrix} F_{p_1}(p_0, z_0, x_0) & 1 & & \\ \vdots & & \ddots & \\ & & & 1 \\ F_{p_n}(p_0, z_0, x_0) & 0 & \cdots & 0 \end{pmatrix}$$

$$|\det DX(0, x_0)| = F_{p_n}(p_0, z_0, x_0) \neq 0$$

so inverse fcn thm applies.

Thm: Assume $f \in C^1(U; \mathbb{R}^n)$ and $|\det Df(x_0)| \neq 0$. Then there exists an open set $V \subset U$ w/ $x_0 \in V$, and $W \subseteq \mathbb{R}^n$ w/ $z_0 = f(x_0) \in W$ s.t.

(1) $f: V \to W$ is 1-1 and onto

(2) $f^{-1}: W \to V$ is C^1

(3) if $f \in C^k$ then $f^{-1} \in C^k$

Example

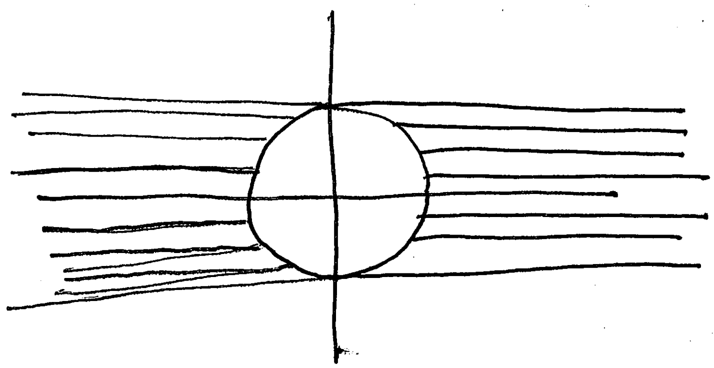
$$\begin{cases} \partial_{x_1} u = 0 & \text{in} \quad \mathbb{R}^2 \setminus B_1(0,1) \\ u = g & \text{on} \quad \partial B(0,1) \end{cases}$$

$$F(p, z, x) = p_2$$

so

$$\begin{cases} \dot{X} = e_1 \\ \dot{z} = 0 \\ \dot{p} = 0 \end{cases} \qquad \begin{cases} X(0, x_0) = x_0 \\ \text{irrelevant} \quad z(0, x_0) = g(x_0) \end{cases}$$

$$X(t, x_0) = x_0 + t e_1$$



all bdy points are non-characteristic except

$\pm e_2$ where $\nu = \pm e_2$ respectively

so $\nu \cdot F_p(p_0, z_0, x_0) = e_1 \cdot \nu = 0$

iff $x_0 = \pm e_2$

otherwise ~~$u(x) =$~~

$u(x) = g(x_0) = g(x_1 - t, x_2)$ where t s.t.

$(x_1 - t)^2 + x_2^2 = 1$ (if $x_1 > 0$)

$$(x_1 - t)^2 = 1 - x_2^2 \qquad \text{for} \qquad -1 \le x_2 \le 1$$

$$x_1 - t = \pm\sqrt{1-x_2^2}$$

$$t = x_1 \mp \sqrt{1-x_2^2}$$

$$u(x) = g\left(x_1 - \operatorname{sgn}(x_1)\sqrt{1-x_2^2},\ x_2\right)$$

Example Eikonal equation.

$$\begin{cases} |Du| = 1 & \text{in } \mathbb{R}^2 \setminus B(0,1) \\ u = \cancel{\cos}\ \cancel{y}\ x_2 & \text{on } \partial B(0,1) \end{cases}$$

$$\begin{cases} \dot{p} = 0 \\ \dot{z} = p \cdot D_pF(p,z,x) = p \cdot \frac{p}{|p|} = |p| \\ \dot{X} = \frac{p}{|p|} \end{cases}$$

w/

$$\begin{cases} X(0) = (\cos\theta, \sin\theta) \\ z(0) = \sin\theta \\ \cancel{p(0) =}\ p(0) = p_n^0 (\cos\theta, \sin\theta) \\ \qquad + p_t^0(-\sin\theta, \cos\theta) \end{cases}$$

we can compute p_t by

$$p_t^0 = Dg \cdot \tau = e_2 \cdot (-\sin\theta, \cos\theta) = \cos\theta$$

then using $|p(0)| = 1$ from the PDE

we get $\quad p_n^0 = \sqrt{1-\cos^2\theta} = \pm\sin\theta$

(note multiple solutions) ~~choose +, this only affects time direction of ODE for Φ.~~

$v \cdot F_{p}(p^0, z_0, x_0) = \frac{p_n^0}{|p|} = p_n^0 = \pm\sin\theta$

so noncharacteristic condition holds for

$\theta \neq 0$ or π.

$p(t) = \cancel{\sin\theta}\ p^0$ ~~with~~

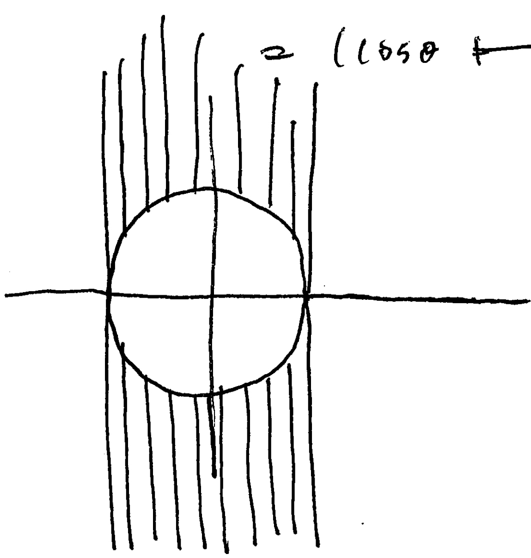
$z(t) = \sin\theta + |p_0| t = \sin\theta + t$

$\Phi(t) = (\cos\theta, \sin\theta) + p_0 t = (\cos\theta, \sin\theta) + (\cos\theta, \sin\theta) \overset{\sin\theta}{\cancel{\cos\theta}}\, t$

$+ (-\sin\theta, \cos\theta) \overset{\cos\theta}{\cancel{\sin\theta}}\, t$

$= (\cos\theta + \cancel{(\cos^2\theta - \sin^2\theta)t}, \sin\theta + \cancel{2\sin\theta\cos\theta t})$

$= (\cos\theta \cancel{+ \cos(2\theta) t}, \sin\theta + \cancel{\sin(2\theta)} t)$



Conservation Laws

We consider the IVP for the scalar conservation law

$$(1) \begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = g(x) & \text{in } \mathbb{R} \end{cases}$$

Recall, after writing the equation as

$$u_t + F'(u) u_x = 0 \quad \text{in } \mathbb{R} \times (0,\infty)$$

Characteristics are

$$\dot{x} = F'(u(\mathbf{x}(t),t)) = F'(g(x_0))$$

since $\dot{z} = \frac{d}{dt} u(\mathbf{x}(t),t) = 0$.

In general the characteristics will cross leading ~~not all of~~ ~~which are not smooth~~ to multiple possibly singular solutions.

• Weak solutions

Suppose to start that $\begin{cases} u_t + F(u)_x = 0 \\ u(x,0) = g(x) \end{cases}$,

u is a smooth solution.

Let $v: \mathbb{R} \times [0,\infty) \to \mathbb{R}$ smooth

w/ cpt support

we call v a test function (like when

we introduced distributions)

multiply the PDE by v and integrate by parts

$$0 = \int_0^\infty \int_{\mathbb{R}} (u_t + F(u)_x) v \, dx \, dt$$

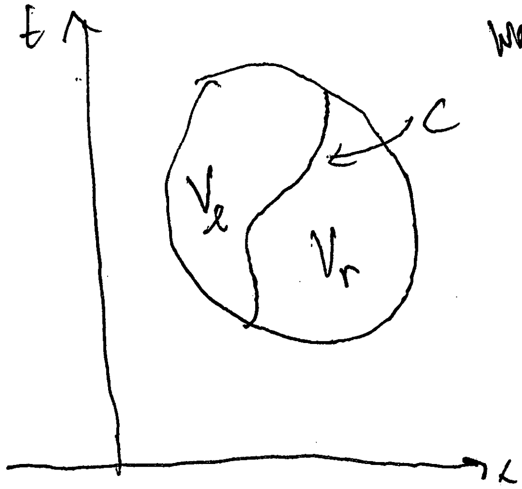
$$\text{IBP} = -\int_0^\infty \int_{\mathbb{R}} u v_t + F(u) v_x \, dx dt - \int_{\mathbb{R}} u(x,0) v(x,0) \, dx$$

using $u(x,0) = g(x)$ we get

(2) $$\boxed{\int_0^\infty \int_{\mathbb{R}} u v_t + F(u) v_x \, dx dt + \int_{\mathbb{R}} g v|_{t=0} \, dx = 0 \quad \text{for all test functions } v}$$

Def: We say $u \in L^\infty(\mathbb{R} \times (0,\infty))$ is an <u>integral solution</u> of (1) if (2) holds for every test functions what kind of non-smooth solutions can we have?

We consider a very particular kind of discontinuous u



We suppose that u is an integral solution of (1) in $V \subseteq \mathbb{R}\times(0,\infty)$ with u smooth ~~in V~~ on either side of a smooth curve C parametrized by ~~[illegible]~~. $(\gamma(t), t)$

V_ℓ is the part of V to the left of C

V_r " " " " " to the right of C

And we suppose u smooth in V_ℓ and V_r, and u, u_t, u_x unif cts in V_ℓ, V_r.

Choosing v w/ cpct support in V_ℓ

$$0 = \int_0^\infty \int_{-\infty}^\infty u v_t + F(u) v_x \, dx\, dt = -\int_0^\infty \int_{-\infty}^\infty (u_t + F(u)_x) v \, dx \, dt$$

$\uparrow$

since u smooth in V_ℓ and v cpctly supported in V_ℓ.

$\Rightarrow$ $u_t + F(u)_x = 0$ ptwise in V_ℓ

same result in V_r

now let v w/ cpct support in V, not necessarily vanishing on C. Then

$$0 = \int_0^\infty \int_{\mathbb{R}} u v_t + F(u) v_x \, dx\, dt$$

$$= \iint_{V_\ell} u v_t + F(u) v_x \, dx\, dt + \iint_{V_r} u v_t + F(u) v_x \, dx\, dt$$

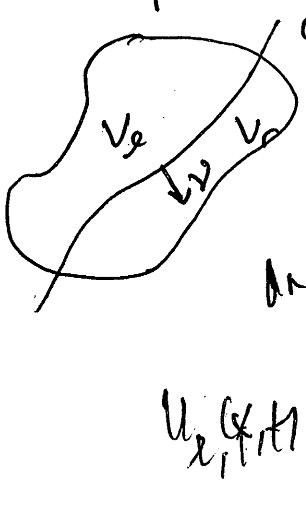
since v cptly supported in V

$$\iint_{V_\ell} u v_t + F(u) v_x = - \iint_{V_\ell} [u_t + F(u)_x] v \, dx\, dt$$

$$+ \int_C [u_\ell \nu_t + F(u_\ell) \nu_x] v \, ds$$

$$= \int_C v (u_\ell \nu_t + F(u_\ell) \nu_x) \, ds$$

where ν is unit normal to C pointing from V_ℓ into V_R, $\nu = (\nu_x, \nu_t)$



~~and $u_\ell(x) = \lim_{\substack{y \to x \\ y \in V_\ell}}$~~

and $u_\ell(x,t)$ is the "left limit"

$$u_\ell(x,t) = \lim_{V_\ell \ni (y,s) \to (x,t)} u(y,s)$$

Similarly

$$\iint_{V_r} u v_t + F(u) v_x \, dx dt = -\int_C [u_r \nu_t + F(u_r)\nu_x] v \, ds$$

so

$$\int_C [(u_\ell - u_r)\nu_t + (F(u_\ell) - F(u_r))\nu_x] v \, ds = 0$$

for every v a smooth cpct support in V.

$$\Rightarrow \quad (u_\ell - u_r)\nu_t + (F(u_\ell) - F(u_r))\nu_x = 0$$

pointwise on C.

now using the parametrization

$$C = \{ (\gamma(t), t) : t \in I \}$$

unit tangent vector is $\dfrac{(\dot\gamma, 1)}{\sqrt{1+|\dot\gamma|^2}}$

unit normal is $\dfrac{(-1, \dot\gamma)}{\sqrt{1+|\dot\gamma|^2}}$

~~Quasi-linear equations~~

~~(linear in p)~~

~~also just need equations for x, z~~

~~not p.~~

~~Fully Nonlinear (nonlinear in p)~~

~~now need full method of characteristics.~~

~~example~~

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so $$F(u_l) - F(u_r) = \dot{\gamma}(t)(u_l - u_r)$$

or $$\dot{\gamma} = \frac{F(u_l) - F(u_r)}{u_l - u_r}$$

we call $[u] = u_l - u_r$ = jump of u across curve C

$[F(u)] = F(u_l) - F(u_r)$ = jump of $F(u)$

$\sigma = \dot{\gamma}$ = speed of curve C

then $$\boxed{\sigma = \frac{[F(u)]}{[u]}}$$ Rankine–Hugoniot condition