first order Equations : Method of Characteristics

We look first at a slightly more general class of (nonlinear) first order equations

$$(1) \quad \begin{cases} u_t + c(x,t,u)u_x = r(x,t,u) & t>0 \\ u(x,0) = f(x) \end{cases}$$

These equations are called quasi-linear since the equation is linear in the highest order derivatives (u_t, u_x).

As before we can interpret as a kind of transport equation w/ reaction term $r(x,t,u)$ and velocity field $c(x,t,u)$.

$$(1') \quad \begin{cases} u_t + c(x,t,u)\cdot\nabla_x u = r(x,t,u) & \text{in } t>0,\ x\in\mathbb{R}^n \\ u(x,0) = f(x) & \text{in } \mathbb{R}^n \end{cases}$$

similar arguments work in higher dimension

We expect that method of characteristics will still work

look at $u(X(t),t)$ along curve $X(t),t))|_{t>0}$

$$\frac{d}{dt} u(X(t),t) = u_t + \dot{X} u_x = r(X(t), t, u(X(t),t))$$

if $\dot{X}(t) = c(X(t), t, u(X(t),t))$

Now characteristic ODE is a system of 2 ODE's for

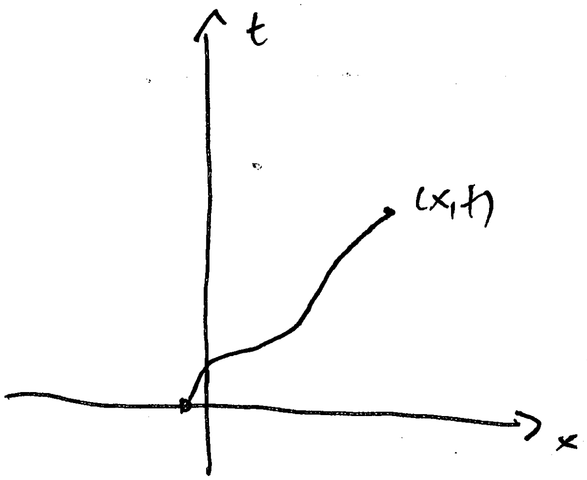
$Z(t) = u(X(t),t)$ and $X(t)$

$$\begin{cases} \dot{X} = c(X,t,Z) \\ X(0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{Z} = r(X,t,Z) \\ Z(0) = u(x_0,0) = f(x_0) \end{cases}$$

It can be useful to keep track of the dependence of the characteristic curve on it's initial point:

$X(t;x_0)$, $Z(t;x_0)$

Now given the solution of the characteristic system



given a point (x,t), $t>0$ we want to find ~~the~~ unique characteristic parametrized ~~$X(t;x_0)$~~ by x_0 so that

$$(x,t) = (X(t;x_0),t)$$ and then

the PDE guarantees

$$u(x,t) = u(X(t;x_0),t) = z(t;x_0)$$

this works ~~if~~ as long as

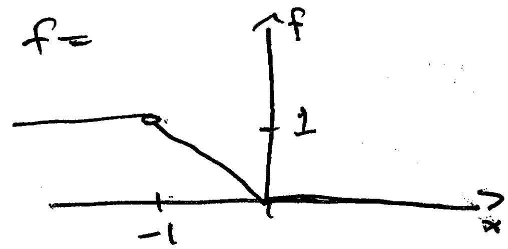
$X(t;x_0) : \mathbb{R} \Rightarrow \mathbb{R}$ is one-to-one and onto $\mathbb{R}$

↳ each (x,t) is on at most one characteristic

↳ each (x,t) is on at least one characteristic example (Burger's Equation)

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = f(x) \end{cases}$$

w/ f =



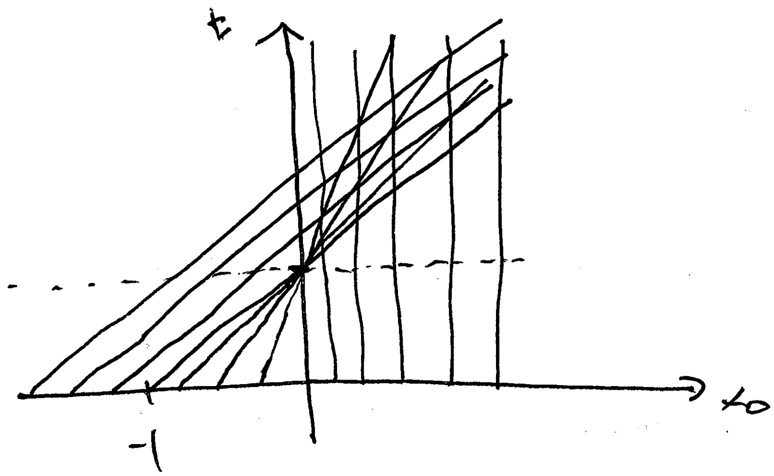
Characteristic system

$$\begin{cases} \dot{X} = Z \\ X(0;x_0) = x_0 \end{cases} \quad \text{w/} \quad \underbrace{\begin{cases} \dot{Z} = 0 \\ Z(0;x_0) = f(x_0) \end{cases}}$$

$$\Downarrow$$

$$Z(t;x_0) = f(x_0)$$

so $\quad X(t;x_0) = x_0 + f(x_0)\,t$



$x_0 \geq 0 \Rightarrow f(x_0) = 0$

$\Rightarrow X(t;x_0) = x_0$

$x \leq -1 \Rightarrow f(x_0) = 1$

$\Rightarrow X(t;x_0) = x_0 + t$

$-1 < x < 0 \Rightarrow f(x_0) = -x_0$

$X(t;x_0) = x_0(1-t)$

characteristics cross for $t \geq 1$, $x \geq 0$

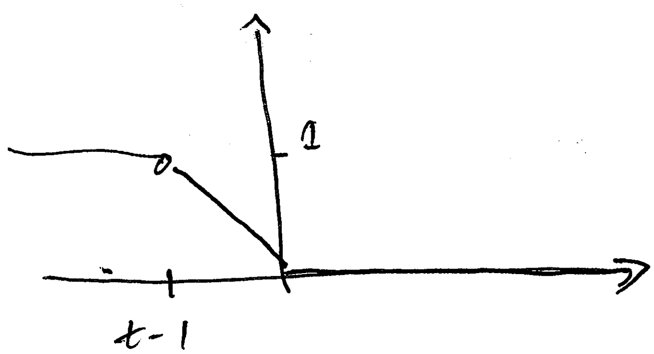
i.e. $X(t;\,\cdot\,)$ is not 1-1 for $t \geq 1$.

for $t \le 1$ we can still invert

$$X = x_0(1-t) \quad \text{for} \quad t-1 < x < 0$$

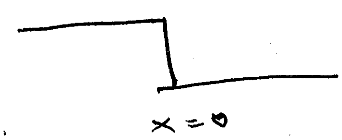
$$x_0 = \frac{x}{1-t} \qquad u(x,t) = u(x_0,0) = -x_0 = -\frac{x}{1-t}$$

$$u(x,t) = \begin{cases} 1 & x \le t-1 \\ -\frac{x}{1-t} & t-1 < x < 0 \\ 0 & x \ge 0 \end{cases}$$



at time $0 < t < 1$

slope steepens till at $t = 1$

$u =$ 

jump.

Quasi-linear equations in $\mathbb{R}^2$ (no time variables)

previous equations we considered all had a time variable which we treated differently than space variables

$$\cancel{a(x,u)\cdot\nabla u = c(x,u) \text{ in}}$$

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

instead of putting initial conditions we put boundary conditions on a curve

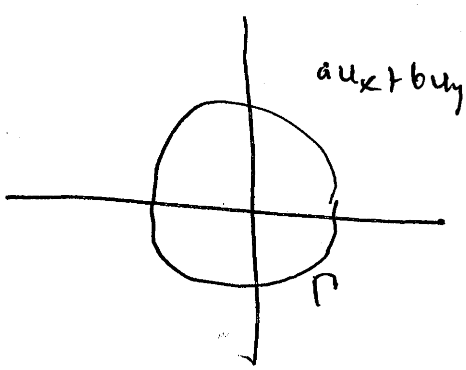
$u = z_0(s)$ on parametrized curve

γ : $x = x_0(s)$, $y = y_0(s)$

e.g. $u = x+y$ on boundary of unit circle in $\mathbb{R}^2$.

$\gamma(s) = (\cos s, \sin s)$ $\quad z_0(s) = \cos(s) + \sin(s)$

$a u_x + b u_y = c$ on $\mathbb{R}^2 \setminus \Gamma$.



~~characteristic equations~~

~~$X(t, s) =$~~

General Method of Characteristics

$$(0) \quad \begin{cases} F(Du, u, x) = 0 & \text{in } U \subseteq \mathbb{R}^n \\ u = g & \text{on } \Gamma \subseteq \partial U \end{cases}$$

Γ, g given

F, g smooth.

As before we look for special curves

$\mathbf{x}(\tau)$, $\tau \in \mathbb{R}$ a parameter

Call $z(\tau) = u(\mathbf{x}(\tau))$

and also $p(\tau) = Du(\mathbf{x}(\tau))$

i.e. $p_i(\tau) = u_{x_i}(\mathbf{x}(\tau))$

for general fully nonlinear equations we will need to keep track of $\mathbf{x}(\tau), z(\tau), p(\tau)$ $2n+1$ scalar quantities

We will need to choose $\mathbf{x}$ in a way so that we can keep track of z, p. i.e. values of u, Du along $\mathbf{x}$.

$$\dot{p}_i(\tau) = \sum_{j=1}^{n} D^2_{x_i x_j} u(\mathbf{X}(\tau))\dot{\mathbf{X}}_j(\tau)$$

not immediately clear what to do w/ this differential on PDE to learn something about $D^2_{x_i x_j} u$.

$\frac{\partial}{\partial x_i}$ applied to PDE $F(p, z, x)$

$$\sum_j \frac{\partial F}{\partial p_j}(Du, u, x)\, u_{x_i x_j} + \frac{\partial F}{\partial z}(Du, u, x)\, u_{x_i} + \frac{\partial F}{\partial x_i}(Du, u, x) = 0$$

if we set (1) $\dot{\mathbf{X}}_j = D_{p_j} F(p(\tau), z(\tau), \mathbf{X}(\tau))$ then

so get (2) $\dot{p}_i = - D_z F(p(\tau), z(\tau), \mathbf{X}(\tau))\, p_i(\tau)$

$$- D_{x_i} F(p(\tau), z(\tau), \mathbf{X}(\tau))$$

and (3) $\dot{z} = Du(\mathbf{X}(\tau)) \cdot \dot{\mathbf{X}} = p(\tau) \cdot D_p F(p(\tau), z(\tau), \mathbf{X}(\tau))$

So we get a system of $2n+1$ ODEs

$$(*) \quad \begin{cases} \dot{p} = - D_x F(p, z, \mathbf{X}) - D_z F(p, z, \mathbf{X})\, p \\ \dot{z} = p \cdot D_p F(p, z, \mathbf{X}) \\ \dot{\mathbf{X}} = D_p F(p, z, \mathbf{X}) \end{cases}$$

and we have proven

Thm If u is a C^2 solution of (0) in U, ~~then~~

Then if $\mathbf{x}(\cdot)$ solves (1) then

$p(\cdot) = Du(\mathbf{x}(\cdot))$ solves (2)

$z(\cdot) = u(\mathbf{x}(\cdot))$ solves (3)

for (τ) s.t. $\mathbf{x}(\tau) \in U$.

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example 1: linear 1st order eqn

~~$F(Du,u,x) = b(x)\cdot Du + c(x)u = 0$~~

$$\begin{cases} F(Du,u,x) = b(x)\cdot Du + c(x)u = 0 & \text{in } U \\ + \text{ BC} \end{cases}$$

so $F(p,z,x) = b(x)\cdot p + c(x)z$

$D_pF = b(x)$, $D_zF = c(x)$ ~~D_xF~~

so $\dot{\mathbf{x}} = D_pF = b(\mathbf{x})$, doesn't involve z or p.

$\dot{z} = p\cdot D_pF = b(x)\cdot p = -c(x)z$

using the PDE.

so we can solve for $\mathbb{X}$, z without using p. This worked because the equation was <u>linear</u> in p.

~~$F(p, z, x) = D_p F$~~

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interpretation for an equation w/ a time

<u>example 1'</u> $\quad u_t + b(x) \cdot D_x u + c(x,t) u = 0$

Characteristics are curves $(\mathbb{X}(\tau), t(\tau))$ in $\mathbb{R}^{n+1}$,

$$F(p, z, x) = F(p_t, p_x, z, x)$$

$$F(p_t, p_x, z, x) = p_t + b(x) \cdot p_x + c(x,t) z$$

$\dot{t} = D_{p_t} F = 1$ $\quad$ so t and τ are the same up to a constant

$\dot{\mathbb{X}} = b(\mathbb{X})$

$\dot{z} = c(\mathbb{X}(\tau), t(\tau))$

Example 2

$$\begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u & \text{in } U \\ u = g & \text{on } \Gamma \end{cases}$$

with $U = \{x_1 > 0,\ x_2 > 0\}$

$\Gamma = \{x_1 > 0,\ x_2 = 0\}$

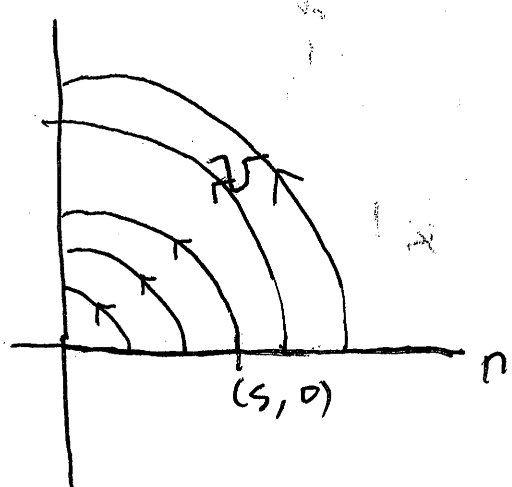
this is a linear equation of form

$$b(x) \cdot Du + c(x) u = 0$$

w/ $b(x) = (-x_2, x_1)$

$c(x) = -1$

so
$$\begin{cases} \dot{X}_1 = -X_2 \\ \dot{X}_2 = X_1 \\ X_2(0) = 0 \\ X_1(0) = s > 0 \end{cases} \qquad \begin{cases} \dot{z} = z \\ z(0) = g(s, 0) \end{cases}$$



$X_1(\tau) = s \cos \tau$

$X_2(\tau) = s \cdot \sin \tau$

$z(\tau) = g(s, 0) e^{\tau}$

This is why we didn't specify boundary data on $x_1 = 0$, $x_2 > 0$ part of ∂V.

Characteristics leave V through that part of ∂V, values will be specified by the equation there.

(although we could have assigned data on vertical (instead of horizontal) axis)

~~how we can invert~~

$$x = (x_1, x_2) = s(\cos\tau, \sin\tau)$$

need to choose of course

$$s = (x_1^2 + x_2^2)^{1/2} = |x|$$

to find τ, $\dfrac{x_2}{x_1} = \dfrac{\sin\tau}{\cos\tau} = \tan\tau$

$$\tau = \arctan\left(\frac{x_2}{x_1}\right)$$ i.e. angle of (x_1, x_2) w/ x axis

$$\boxed{u(x) = g(s,0)\, e^{\tau} = g(|x|, 0)\, e^{\arctan\left(\frac{x_2}{x_1}\right)}}$$

Non-characteristic Boundary data

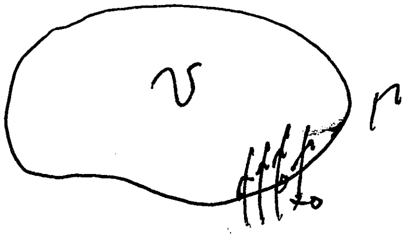
As we saw in the previous example

need characteristics to be flowing <u>into</u>, or out of U at $x_0 \in \partial U$ if we want to put boundary data there.

i.e. need at x_0

$$\boxed{\dot{X} = D_p F(p_0, z_0, x_0) \cdot \underbrace{\nu(x_0)}_{\text{outward normal to } U \text{ at } x_0} \neq 0}$$

Called non-characteristic condition.



$x_0 \in \Gamma$

from ODE theory we can solve the system

$$\begin{cases} \dot{X} = D_p F(p, z, X) \\ \dot{z} = p \cdot D_p F(p, z, X) \\ \dot{p} = -D_x F(p, z, X) - p D_z F(p, z, x) \end{cases}$$

except we know

$$X(0) = x_0 \in \Gamma$$

$$z(0) = u(x_0) = g(x_0) \quad \text{bdry data}$$

$$p(0) \cdot e = Du(x_0) \cdot e = Dg(x_0) \cdot e$$

if e tangent to Γ at x_0

but we don't know $p(0) \cdot \nu(x_0)$.

We only have the equation

$$F(p(0; x_0), z(0; x_0), X(0; x_0)) = 0$$

need to solve this for

$$p(0) \cdot \nu(x_0)$$

parametrize nbhd of x_0 by $\Phi: \underset{D \subseteq \mathbb{R}^{n-1}}{\cancel{\mathbb{R}^{n-1}}} \to \Gamma$

$\Phi(0) = x_0$. $\Phi(s) = x \in \Gamma$ near x_0

want the map

$\cancel{X}$ $(t,s) \mapsto X(t,s)$ to be invertible

by inverse function thm this is true in a

nbhd of $(0,0)$ if

$DX(0,0)$ non-degenerate

$$DX(0,0) = \left(\frac{\partial X}{\partial t} \middle| \; \underbrace{D_s X(0,0)}_{\text{tangent vectors}} \right)$$

$X(0,s)$ $= \Phi(s)$ is a parametrization of Γ

$D\Phi(0)$ is $n \times (n-1)$ matrix w/

$(n-1)$ linearly indep columns

spanning tangent space to Γ at x_0

i.e. $\{ w : w \cdot \nu(x_0) = 0 \}$

as long as $\left.\frac{\partial X}{\partial t}\right|_{(0,0)} = F_p(p(0), z(0), X(0))$ is not

tangential columns will be linearly independent

$\Rightarrow$ OK.

example: Eikonal Equation (equation of the distance function to a set)

$$\begin{cases} |Du|^2 = 1 & \text{in } \mathbb{R}^2 \setminus B(0,1) \\ u = 0 & \text{on } \partial B(0,1) \end{cases}$$

$F(p,z,x) = |p|^2 - 1$

$F_p = 2p$

$F_z = 0$

$F_x = 0$

so $\begin{cases} \dot{x} = 2p \\ \dot{z} = 2|p|^2 \\ \dot{p} = 0 \end{cases}$

w/ $\begin{cases} x(0) = (\cos\theta, \sin\theta) \\ z(0) = 0 \\ p(0) \cdot (-\sin\theta, \cos\theta) = 0 \\ |p(0)| = 1 \end{cases}$

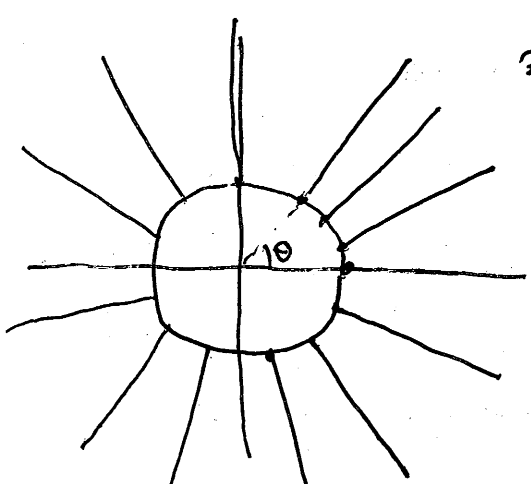
so using the last two conditions we see

$p(0) = (\cos\theta, \sin\theta) = x(0)$

and since $\dot{p} = 0$, $p(t) = x(0)$

so $x(t) = x(0) + 2x(0)t = (1+2t)(\cos\theta, \sin\theta)$

$z(t) = 0 + 2|p|^2 t = 2t$



so for $x \in \mathbb{R}^n \setminus B(0,1)$

want to invert

$(\theta, t) \to x(t;\theta)$

$x = |x|(\cos\theta, \sin\theta)$ so $1+2t = |x|$

and $t = \frac{|x|-1}{2}$ so

$$\boxed{u(x) = 2t = |x| - 1}.$$