Wave Equation in $\mathbb{R}^2$ and $\mathbb{R}^3$: Huygens Principle

in $\mathbb{R}^3$, let u be a smooth solution of wave eqn

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ u(x,0) = \phi(x) & u_t(x,0) = \psi(x) \end{cases}$$

We derive an equation for the spherical means of u which, in odd dimensions, reduces to a 1-d wave equation.

$x \in \mathbb{R}^n$, $t > 0$, $r > 0$ define

$$U(x;t,r) = \fint_{\partial B(x,r)} u(y,t)\, dS_y$$

$$\Phi(x;r) = \fint_{\partial B(x,r)} \phi(y)\, dS_y$$

$$\Psi(x;r) = \fint_{\partial B(x,r)} \psi(y)\, dS_y$$

Lem fix $x \in \mathbb{R}^3$ and let U, Φ, Ψ as above then

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ U = \Phi, \ U_t = \Psi & \mathbb{R}_+ \times \{t=0\} \end{cases}$$

proof: we computed before in proof of the mean value property that

$$v_r(x;r,t) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)\, dy$$

thus $\lim_{r \to 0} v_r(x;r,t) = 0$.

taking another derivative in r

$$v_{rr}(x;r,t) = \frac{1}{n} \fint_{B(x,r)} \Delta u(y,t)\, dy + \frac{r}{n} \frac{d}{dr} \fint_{B(0,1)} \Delta u(x+rz,t)\, dz$$

something wrong?

$$= \frac{1}{n} \fint_{B(x,r)} \Delta u(y,t)\, dy + \frac{r}{n} \fint_{B(0,1)} z \cdot D(\Delta u)(x+rz,t)\, dz$$

$$= \frac{1}{n} \fint_{B(x,r)} \Delta u(y,t)\, dy + \frac{r}{n} \frac{1}{|B_1|} \int_{\partial B(0,1)} \Delta u(x+rz,t)\, dS_z - r \fint_{B(0,1)} \Delta u(x+rz,t)\, dz$$

etc to show $\frac{\partial^k}{\partial r^k} v$ is continuous if $u \in C^k$.

Notice

$$U_{tt} - \underbrace{v_{rr} - \frac{n-1}{r} v_r}_{} = 0$$

radial part of Laplacian, same as

$$\frac{1}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r v \right)$$

$$\left(r^{n-1} \mathcal{U}_r\right)_r = \left(\frac{1}{n\alpha(n)} \int_{B(x,r)} \Delta u(y,t)\,dy\right)_r$$

$(F(0) = 0)$

$F(0) + \int_0^r F'(s)\,ds$

$\quad\quad \parallel$

$$= \left(\frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt}(y,t)\,dy\right)_r$$

since $\int_0^r \int_{\partial B(x,s)} f(y)\,dS_y\,ds$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}(y,t)\,dS_y$$

$= \int_{B(x,r)} f(y)\,dy$

$$= r^{n-1} ⨍_{\partial B(x,r)} u_{tt}(y,t)\,dS_y = r^{n-1}\,\mathcal{U}_{tt}$$

$= F(r)$

so $$\mathcal{U}_{tt} - \frac{1}{r^{n-1}}\left(r^{n-1}\mathcal{U}_r\right)_r = 0 \quad \text{in } \mathbb{R}_+ \times (0,\infty)$$

initial data is by continuity of u.

<u>Now in $n=3$</u>:

Call $V = r\mathcal{U}$ then

$$V_{tt} = r\,\mathcal{U}_{tt} = r\left[\mathcal{U}_{rr} + \frac{2}{r}\mathcal{U}_r\right]$$

$$= r\,\mathcal{U}_{rr} + 2\mathcal{U}_r$$

$$= \left(\mathcal{U} + r\,\mathcal{U}_r\right)_r = V_{rr}$$

so
$$\begin{cases} V_{tt} - V_{rr} = 0 & \text{in } \mathbb{R}_+ \times [0,\infty) \\ V = r\Phi \quad V_t = r\Psi & \text{on } \mathbb{R}_+ \times \{t=0\} \\ V(x;0,t) = 0 \end{cases}$$

Now we can apply D'Alembert's formula ~~to solve for~~ ~~to~~ ~~the~~. w/ $V(x;0,t)=0$ BC to get

$$V(x;r,t)=\frac{1}{2}\left[(r+t)\Phi(x;r+t)-(t-r)\Phi(x;t-r)\right]$$
$$+\frac{1}{2}\int_{t-r}^{t+r}\rho\,\Psi(x;\rho)\,d\rho$$

now since $u(x,t)=\lim_{r\to 0}V(x;r,t)$

$$u(x,t)=\lim_{r\to 0^+}\frac{V(r;x,t)}{r}=\lim_{r\to 0^+}\left[\frac{1}{2}\left[\Phi(x;r+t)-\Phi(x;t-r)\right]+\frac{1}{2}\frac{1}{r}\int_{t-r}^{t+r}\rho\,\Psi(x;\rho)\,d\rho\right]$$

$$=\frac{\partial}{\partial t}(t\,\Phi(x;t))+t\,\Psi(x;t)$$

$$=\frac{\partial}{\partial t}\left(t⨍_{\partial B(x,t)}\phi(y)\,dS_y\right)+t⨍_{\partial B(x,t)}\psi(y)\,dS_y$$

now using $⨍_{\partial B(x,t)}\phi(y)\,dS_y=⨍_{\partial B(x,1)}\phi(x+tz)\,dS_z$

we get
$$\frac{\partial}{\partial t}⨍_{\partial B(x,t)}\phi(y)\,dS_y=⨍_{\partial B(x,1)}z\cdot D\phi(x+tz)\,dS_z$$
$$=⨍_{\partial B(x,t)}\frac{y-x}{t}\cdot D\phi(y)\,dS_y$$

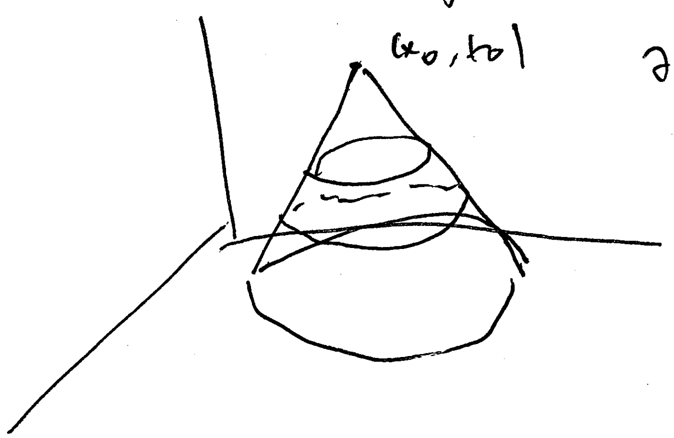
and so

$$\boxed{u(x,t) = \fint_{\partial B(x,t)} \phi(y) + t\psi(y) + D\phi(y)\cdot(y-x)\, dS_y}$$

Kirchoff's formula for wave eqn in $n=3$.

~~$n=2$: we can't tran~~

Notice $u(x,t)$ depends only on the boundary of the light cone



$$\partial \Delta(x_0,t_0) = \{(x,t)\in\mathbb{R}^3\times\mathbb{R}_+ : |x-x_0| = (t_0-t)\}$$

this is really important ~~[illegible]~~ (sound waves)

$\underline{n=2}$: Situation is quite different, still

We can get an explicit formula by using ~~projection method~~ from $\mathbb{R}^3$

<u>method of descent</u>

Consider $\tilde{u}$ solving

$$\begin{cases} \tilde{u}_{tt} - \Delta\tilde{u} = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \tilde{u}(x,0) = \tilde{g}(x) \quad \tilde{u}_t(x,0) = \tilde{\psi}(x) & \text{in } \mathbb{R}^3 \times \{t=0\} \end{cases}$$

~~$\tilde{g}(x_1,x_2,x_3) = g$~~

$\tilde{\phi}(x_1,x_2,x_3) = \phi(x_1,x_2) \qquad \tilde{\psi}(x_1,x_2,x_3) = \psi(x_1,x_2)$

Note: $\tilde{u}^{\ell} = \tilde{u}(x_1,x_2,x_3+\ell,t)$ solves wave eqn

for each fixed $\ell \in \mathbb{R}$

w/ $\tilde{u}^{\ell}(x,0) = \tilde{\phi}(x_1,x_2,x_3) = \phi(x_1,x_2)$

$= \tilde{u}(x,0)$ same for $\tilde{u}^{\ell}_t = \tilde{u}_t$ at $t=0$

so <u>by uniqueness</u> $\tilde{u}^{\ell} = \tilde{u} \quad \forall\, t>0$

$\Rightarrow \quad \tilde{u}(x_1,x_2,0,t)$ solves wave eqn in $\mathbb{R}^2 \times (0,\infty)$

We use the 3-d Kirchoff formula to derive a formula in 2-d.

$$\fint_{\partial\tilde{B}(\tilde{x},t)} \tilde{\phi}\, d\tilde{S}_y = \frac{1}{4\pi t^2}\int_{\partial\tilde{B}(\tilde{x},t)} \tilde{\phi}\, d\tilde{S}$$

$$= \frac{2}{4\pi t^2}\int_{B(x,t)} \phi(y)\left(1+|D\gamma(y)|^2\right)^{1/2} dy$$

w/ $\gamma(y) = (t^2-|y-x|^2)^{1/2}$

so

$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{t}{2}\fint_{B(x,t)} \phi(y)\left(1+|D\gamma(t,y)|^2\right)^{1/2} dy\right)$$

$$+ \frac{t}{2}\fint_{B(x,t)} \psi(y)\left(1+|D\gamma(y)|^2\right)^{1/2} dy$$

$$|D\gamma|^2 = \frac{|x-y|^2}{(t^2-|y-x|^2)} \qquad \left(1+|D\gamma|^2\right)^{1/2} = \frac{t}{(t^2-|y-x|^2)^{1/2}}$$

$$u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2\fint_{B(x,t)} \frac{\phi(y)}{(t^2-|x-y|^2)^{1/2}}dy\right)$$

$$+ \frac{1}{2}t^2\fint_{B(x,t)} \frac{\psi(y)}{(t^2-|x-y|^2)^{1/2}}dy$$

$$t^2 \fint_{B(x,t)} \frac{\phi(y)}{(t^2-|x-y|^2)^{1/2}}\,dy = t \fint_{B(x,1)} \frac{\phi(x+tz)}{(1-|z|^2)^{1/2}}\,dz$$

so $\quad \frac{\partial}{\partial t}\Big(\qquad \Big)$

$$= \fint_{B(x,1)} \frac{\phi(x+tz) + tz\cdot D\phi(x+tz)}{(1-|z|^2)^{1/2}}\,dz$$

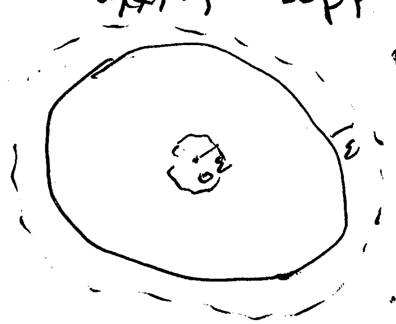
$$= t \fint_{B(x,t)} \frac{\phi(y) + (y-x)\cdot D\phi(y)}{(t^2-|x-y|^2)^{1/2}}\,dy$$

so $\quad u(x,t) = \frac{1}{2}\fint_{B(x,t)} \frac{t\phi(y) + t^2\psi(y) + t(y-x)\cdot D\phi(y)}{(t^2-|y-x|^2)^{1/2}}\,dy$

Things to notice: think of initial data (ϕ,ψ) localized in $B(0,\varepsilon)$

In $\mathbb{R}^2$

$u(x,t)$ supported in $B(0,t+\varepsilon)$



In $\mathbb{R}^3$

$u(x,t)$ supported in annulus of width ε.