Basic Theory of PDE

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Office Hours:

Text: "Partial Differential Eqns: an introduction to theory & Applications" Shearer & Levy

Grading: 40% final
30% midterm
30% Homework

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Problem Session:

Course Requirements: (1) Multi-variable Calculus
(2) ODE's (see reading assignment)

~~PDE~~

Introduction

PDE describe many, many, many physical systems like

- diffusion of solvent / tracer
- kinetic equations of particle density in diffuse gas
- brownian particles
- vibrating string, drumhead, metal plate
- evolution of wave functions in Quantum Mechanics
- fluid flow
- water droplet shapes
- ice melting in water

typically a PDE problem ~~is set in~~

is concerned w/ an unknown function u (r.v. or vector valued)

defined on an open set $U \subseteq \mathbb{R}^n$.

$x = (x_1, \dots, x_n)$ are points in $\mathbb{R}^n$

for many equations one of the independent variables

has a natural interpretation as a time

so we would consider $u(x,t)$

domain $U \times [0,\infty)$ or $U \times (-\infty,\infty)$

~~The test~~

A PDE is an equation (or system of equations)

$$\begin{cases} F(D^k u, D^{k-1} u, \ldots, Du, u, x) = 0 & \text{in } U \\ \text{boundary conditions on } \partial U. \end{cases}$$

the <u>order</u> of the PDE is k the highest order derivative involved in the equation.

(1) >>>

<u>Transport Equation</u>: $\begin{cases} u_t + c u_x = 0 & \text{on } \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & \text{on } \mathbb{R} \end{cases}$

more generally a transport equation in $\mathbb{R}^n$

$$\begin{cases} u_t + c(x)\cdot \nabla_x u = 0 & \text{on } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x) & \text{on } \mathbb{R}^n \end{cases}$$

Modelsq advection of a tracer particle by a flow field $c(x)$.

<u>Laplace Equation</u>: $-\Delta u = 0$ or $-\Delta u = f(x)$
(Poisson's Equation)

where $\Delta u = \nabla^2 u = \nabla\cdot(\nabla u) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$

- models equilibria of diffusion processes
- Newtonian gravitation
- Electrostatics
- Complex Analysis
- Associated w/ SRW and Brownian Motion

Heat Equation: $u_t - \Delta u = 0$

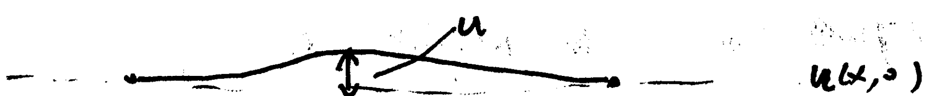
- Models diffusion processes, u is concentration of diffusant
- Associated w/ SRW and Brownian Motion

Wave Equation: $\cancel{u_{tt} - c^2 \Delta u = 0}$

$$u_{tt} - \Delta u = 0$$

Models wave propagation, u is local displacement from equilibrium.

Think of a guitar string



(Schrödinger Eqn):

These three 2nd order equations represent 3 important classes of equation

elliptic	parabolic	hyperbolic
$-\Delta u = 0$	$u_t - \Delta u = 0$	$u_{tt} - \Delta u = 0$
"energy minimizer" ~~or critical point~~	"energy dissipating"	"energy conserving"

We will carefully study

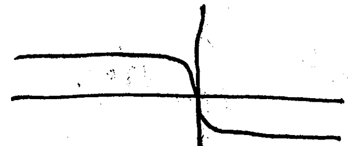
paradigmatic of 2nd order eqns
- Laplace
- Heat
- Wave

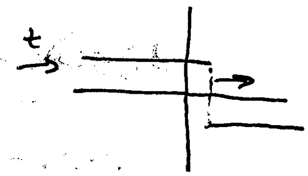
- First order equations

More PDE (nonlinear): (These will test our understanding of what it means to be a solution of a PDE)

Burger's Eqn:

$$u_t + uu_x = 0$$





Like a transport equation but propagation speed depends on the height u.

This equation has shocks

solutions typically steepen and develop a discontinuity in finite time.

Fisher's Eqn $\quad u_t = \Delta u + f(u)$

$f(u) = u(1-u)$ solutions w/ $u > 0$ are interpreted as population density.

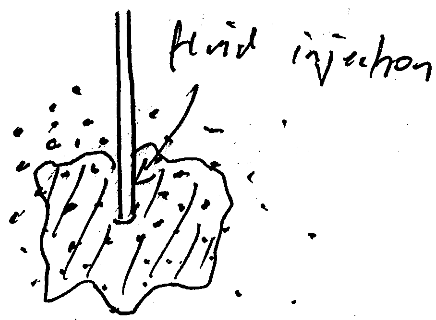
$f(u)$ models environmental limitations on pop growth

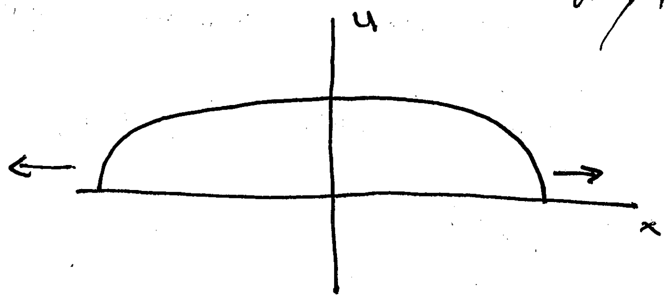
Δu models diffusive spreading of population.

— travelling wave solutions specify the spreading speed.

Porous Medium Equation $\quad u_t - \Delta(u^m) = 0$

model for fluid flow in sand $\quad$ [special spreading solution governs long time behavior





Schrödinger Equation $\quad iu_t - \Delta u = 0$

$u : \mathbb{R}^d \to \mathbb{C}$ wave function

$|u|^2$ probability density

Navier-Stokes Equations

$u \in \mathbb{R}^3$ fluid velocity, incompressible

$$\begin{cases} \underbrace{u_t + u\cdot\nabla u}_{\text{transport/Burgers}} = -\nabla p + \underbrace{\nu \Delta u}_{\text{diffusion}} & \text{] conservation of momentum} \\ \nabla \cdot u = 0 \end{cases}$$

∇p enforces incompressibility condition

ν viscosity $\quad$ (+ Euler Equations)

famous open problem well-posedness of Navier-Stokes

(1) Solutions of PDE and Boundary/initial Conditions

A solution of a PDE is a r.v. function u k-times differentiable everywhere and

$F(D^k u, \dots, Du, u, x) = 0$ pointwise in U.

as we will see / have seen this notion of a solution is often too strong for nonlinear equations and later we will see various notions of weak solutions of PDE.

typically just the PDE is not sufficient to identify unique solution need initial / boundary conditions.

e.g. (ODE) $\dot{X}_t = \kappa X_t$

solutions are $X_t = Ae^{\kappa t}$ for any $A \in \mathbb{R}$

need to specify initial data

$X(0) = X_0$

$X_t = X_0 e^{\kappa t}$ unique solution.

PDE A guitar string, held fixed on both ends

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } (0,1) \times (0,\infty) \\ u(0,t) = u(1,t) = 0 & \\ u(x,0) = f(x) \quad u_t(x,0) = g(x) & \text{in } [0,1] \end{cases}$$

need both boundary conditions and initial conditions.

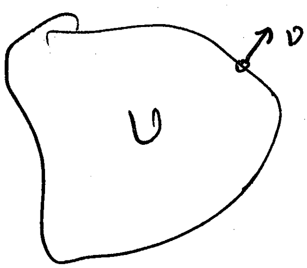
This is usually called <u>Dirichlet Boundary Conditions</u> when value of u is fixed on ∂U.

Can also have <u>Neumann boundary conditions</u>

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } U \times (0,\infty) \\ \boxed{\dfrac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial U \times (0,\infty)} \end{cases}$$

+ Initial data

outward normal derivative



$$\frac{\partial u}{\partial \nu} = Du \cdot \nu$$

<u>Fundamental issues of PDE theory</u>

Well-posedness (in the sense of Hadamard) of a PDE problem w/ BC and/or IC

1. Existence – "is there a solution?"
2. Uniqueness – "Is there only one solution of I/BVP?"
3. Stability – "Does the solution depend continuously on the boundary/initial data/coefficients?"

Additional questions:

Stability * - "does solution depend continuously on the coefficients of the equation?"

4. Regularity - "Do solutions have additional derivatives over what is necessary to be a solution?"

5. Long time / Large Scale Behavior - "What do solutions do?"

Why are these questions fundamental?

~~Phys:~~ ~~Many~~ PDE come from physics, engineering biology, economics

Mathematicians want to understand

- When the derivation makes sense (this can elucidate the ~~difference~~ border between physical regimes where different factors are dominant)
- When you can guarantee the accuracy of a numerical method (this will usually follow from existence + stability)

Derivations of Equations ; Conservation/Balance Laws

Many PDE arise ~~as~~ ~~from conservatio~~
as localized versions of conservation laws
of mass, momentum and energy

typical quantities of interest are density, velocity, stress
and internal energy.

These conservation laws express a physical constraint
that say the total quantity of (say) mass of a certain
in a domain $V \subseteq \mathbb{R}^n$ ~~cannot change~~
can only change by
(i) creation or destruction in the domain V
~~(for example by reaction w/~~
another element, or
birth / death in biological settings)
(ii) flux through ∂V

1. write down the balance law
2. use Divergence thm to relate boundary flux to volume integral over V.
3. Deduce, since V was arbitrary, that the integrand must vanish pointwise.

3. Close the system by relating flux and/or internal creation/destruction of u back to u called constitutive laws. Come from physical considerations. (not necessarily PDEs but could be)

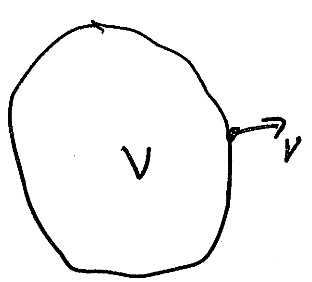
Let V an open subset of U w/ outward normal $\nu(x)$ on ∂V

$$A(t) = \int_V u(x,t)\,dx$$

$$\frac{dA}{dt} = \int_V \frac{\partial u}{\partial t}(x,t)\,dx$$

1. change in A must be through
   (i) flux through ∂V or
   (ii) source/sink in V

$$\frac{dA}{dt} = \underbrace{-\int_{\partial V} Q(x,t)\cdot \nu\, dS}_{\text{flux through } \partial V} + \underbrace{\int_V f(x,t)\,dx}_{\text{source/sink in } V.}$$



$Q \cdot \nu > 0$ means u is flowing out of V

$\Rightarrow A'(t) < 0$

Divergence Theorem

$$\int_V \operatorname{div} Q(x,t)\, dx = \int_{\partial V} Q(x,t) \cdot \nu \, dS$$

(true for any $Q(x)$)

arrive at

$$\int_V \left[\frac{\partial u}{\partial t} + \operatorname{div} Q - f \right] dx = 0 \quad \text{for any } V \subseteq U$$

smooth enough for divergence thm to hold.

$\Rightarrow$ integrand is identically zero in U.

$$\frac{\partial u}{\partial t} + \operatorname{div} Q = f \quad \text{in } U \quad \text{pointwise.}$$

(assuming sufficient regularity)

(3) need to relate flux back to u to close the system. This depends on the particular situation.

example 1: (continuity Equation)

$u(x,t)$ - density of ~~po~~ tracer

~~flo~~ advected by velocity field $V(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$

flux $\quad Q(x,t) = V(x,t)\, u(x,t)$

so $\quad \partial_t u + \cancel{\nabla} \operatorname{div}(V u) = f$.

example 2: (heat equation)

$u(x,t)$ - temperature $\qquad$ energy $e = \rho c u$

ρ density of material, c specific heat

$Q(x,t)$ - heat flux

$F(x,t)$ - source term

Fourier's Law $\qquad Q(x,t) = -\kappa \nabla u$

$\kappa > 0$ called thermal conductivity

energy conservation leads to

$$u_t - k \operatorname{div} \nabla u = \cancel{flux}\ f$$

$$k = \frac{\kappa}{\rho c} \qquad f = \frac{F}{\rho c} .$$

example 3: (wave equation)

~~total~~ momentum is conserved quantity

ρu_t - momentum $\qquad$ (ρ constant density)

$$\frac{\partial}{\partial t}(\rho u_t) + \operatorname{div} Q = 0$$

where $Q(x,t)$ - momentum flux

In elasticity theory

$$Q(x,t) = -k \nabla u$$

leads to wave equation

$$u_{tt} - c^2 \Delta u = 0$$

$$c^2 = \frac{k}{\rho}$$

Method of Characteristics for the Transport Eqn

IVP $\begin{cases} u_t + c u_x = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & \text{on } \mathbb{R} \end{cases}$ (1)

idea: look at $u(x(t),t)$ along special paths $x(t)$

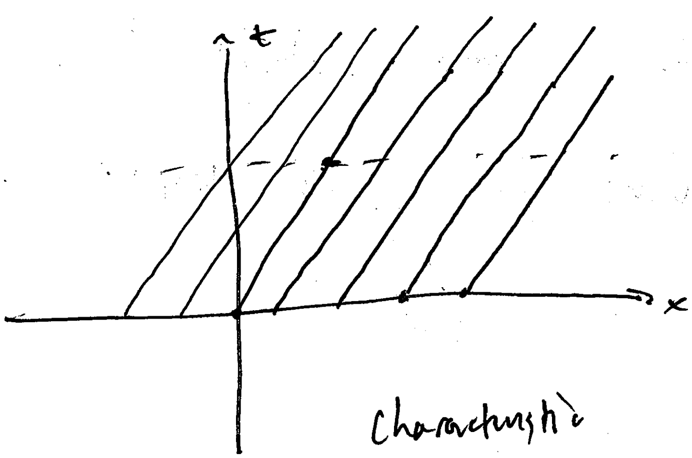
$$\frac{d}{dt} u(x(t),t) = \left(\frac{\partial u}{\partial t} + \dot{x}(t)\frac{\partial u}{\partial x}\right)\Big|_{(x(t),t)}$$

if u is a solution ~~then~~ of (1)

and we choose $\dot{x}(t) = c$

then $\frac{d}{dt}\left[u(x(t),t)\right] = 0$

u is constant along trajectories of



the ODE $\dot{x} = c$

solutions $x(t) = x_0 + ct$

for each (x,t)

~~evaluate~~ find out which

characteristic (x,t) is lying on

i.e. solve $x = x_0 + ct$ for x_0

$x_0 = x - ct$

then $u(x,t) = f(x-ct)$

Note that f does not need to be C^1 to define $u(x,t) = f(x-ct)$ but if it is

then
$$\partial_t u = (-c)f'(x-ct)$$
$$\partial_x u = f'(x-ct)$$
$$\cancel{\partial_t u + c} \quad u_t + cu_x = f'(x-ct) - c f'(x-ct) = 0.$$

This shows existence of a solution.

We also showed uniqueness of C^1 solution since we showed ~~that~~ actually that an arbitrary solution of the IVP had to satisfy $u(x,t) = f(x-ct)$.

Stability clear as well from the explicit formula.

Like ~~most~~ wave type / hyperbolic eqns regularity does not improve for solutions of general transport equation.

We can solve
$$\begin{cases} u_t + c(x) \cdot \nabla_x u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x) & \text{in } \mathbb{R}^n \end{cases}$$
$c : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz cts.

in very similar manner (although solution is not explicit)