MATH 212: Homework 4

Due Tuesday, October 30

Numbered problems are from *Numerical Analysis* by L. Ridgway Scott. Other problems are from *An Introduction to Numerical Analysis* by Süli and Mayers (this is just for proper credit, you shouldn't need to reference that book).

Problem 1. 6.10, 6.12

Problem 2. Recall the condition number of an invertible matrix A, associated with a given operator norm, is defined by $\kappa(A) = \|A\|\|A^{-1}\|$. Call κ_p to be the condition number associated with the $\|\cdot\|_p$ operator norm. Suppose that A is a real invertible matrix, prove that

$$\kappa_2(A) = \left(\frac{\lambda_n}{\lambda_1}\right)^{1/2}$$

where λ_1 and λ_n are respectively the smallest and largest eigenvalues of the matrix A^TA. Show that $\kappa_2(Q) = 1$ for an orthogonal matrix Q. Conversely if $\kappa_2(A) = 1$ show that all eigenvalues of A^TA are equal, and deduce that A is a scalar multiple of an orthogonal matrix.

Problem 3. Let A be an $n \times n$ matrix. Suppose that λ is an eigenvalue of A^TA, show that

$$0 \leq \lambda \leq \|A^T\|\|A\|$$

for any operator norm $\|\cdot\|$. Use this to show that for any nonsingular matrix A

$$\kappa_2(A) \leq \kappa_1(A)^{1/2}\kappa_\infty(A)^{1/2}.$$

Problem 4. Show that if $\|A\| < 1$ then $I - A$ is nonsingular. Then, for $\|A\| < 1$ show the formula

$$(I - A)^{-1} = I + A(I - A)^{-1}.$$

Use this to bound

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

Problem 5.. Let A be a nonsingular $n \times n$ matrix and let $b \in \mathbb{R}^n \setminus \{0\}$. Suppose that $Ax = b$, $(A + \delta A)(x + \delta x) = b$, and $\|A^{-1}\delta A\| < 1$. Use the previous problem to show that

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1}\delta A\|}{1 - \|A^{-1}\delta A\|}.$$