~~Line Integrals revisited~~

Surface Integrals on $\mathbb{R}^n$

$U \subseteq \mathbb{R}^2$ a bdd domain, open, ∂U measure 0

$g: U \to \mathbb{R}^n$ a 1-1 smooth map

$Dg(u)$ has rank 2 $\forall\ u \in U$

then $S = g(U)$ is a *parametrized surface*

for $\omega \in A^2(\mathbb{R}^n)$ define

$$\int_S \omega = \int_U g^* \omega$$

Examples $\boxed{1}$ $D =$ ~~unit disk in $\mathbb{R}^2$~~ $[0,1] \times [0, 2\pi] \subseteq \mathbb{R}^2$

$$g(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{1-r^2})$$

parametrizes upper ~~half~~ hemisphere

$\omega = z\, dx \wedge dy$ $\qquad g^*\omega = \sqrt{1-r^2}\, dg_1 \wedge dg_2$

$\cancel{= \sqrt{1-r^2}\,(\cos\theta\, dr - r\sin\theta\, d\theta) \wedge}$

$$g^*\omega = \sqrt{1-r^2}\,(\cos\theta\, dr - r\sin\theta\, d\theta)\wedge(\sin\theta\, dr + r\cos\theta\, d\theta)$$

$$= \sqrt{1-r^2}\; r\; \cancel{d\theta}\, dr\wedge d\theta$$

so $$\int_S \omega = \int_D g^*\omega = \int_0^1\int_0^{2\pi} \sqrt{1-r^2}\, r\, d\theta\, dr$$

$$= 2\pi \int_0^1 \sqrt{u}\,\frac{du}{2}$$

$$= \cancel{\frac{4\pi}{3}}\ \frac{2\pi}{3}.$$

$\boxed{2}$ $g: (0,2\pi)\times(0,\pi/2) \to \mathbb{R}^3$ spherical coordinates

$$g(\theta,\phi) = (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi)$$

$$g^*\omega = g^*(z\, dx\wedge dy) = \cos\phi\; d(\cos\theta\sin\phi)\wedge d(\sin\theta\sin\phi)$$

$$= \cos\phi\,(-\sin\theta\sin\phi\, d\theta + \cos\theta\cos\phi\, d\phi)\wedge(\cos\theta\sin\phi\, d\theta + \sin\theta\cos\phi\, d\phi)$$

$$= \cos\phi\,(-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi)\, d\theta\wedge d\phi$$

$$= \cancel{\sin\phi}\quad -\sin\phi\cos^2\phi\; d\theta\wedge d\phi$$

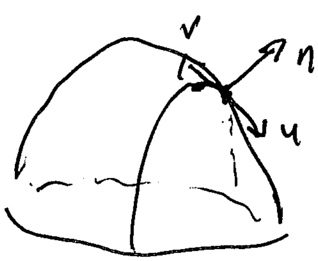
$$\int_S \omega = \int_{(0,2\pi)\times(0,\pi/2)} -\sin\phi\cos^2\phi\; d\theta\wedge d\phi = -2\pi\int_0^{\pi/2} \sin\phi\cos^2\phi\, d\phi$$

$$= -\frac{2\pi}{3}$$

opposite signs? parametrization have different orientations

An <u>orientation</u> (vaguely) is a ctsly & varying notion of what is a positively oriented basis for a surface.

e.g. if we can specify a ctsly varying normal direction n for a 2d surface $S \subseteq \mathbb{R}^3$



then n specifies a "positively oriented" basis by

$$\begin{vmatrix} | & | & | \\ n & u & v \\ | & | & | \end{vmatrix} > 0$$ where

u, v are basis for tangent space

aiming to generalize,
~~more generally~~ ~~$dx_1 \wedge dx_2$ spe~~

note since $\dim \Lambda^2(\mathbb{R}^2)^* = 1$

every element is a nonnegative multiple

of $dx_1 \wedge dx_2$ or $dx_2 \wedge dx_1$

$\phi \in \Lambda^2(\mathbb{R}^2)^*$
~~$dx_1 \wedge dx_2$~~ defines an orientation by

v_1, v_2 are positively oriented if

~~$dx_1 \wedge dx_2$~~ $\phi(v_1, v_2) > 0$.

by analogy, a nowhere zero 2-form

ω on S defines an orientation

by u, v basis for tangent plane at $x \in S$

positively oriented iff $\omega_x(u, v) > 0$.

A C^1 parametrization $g: \mathbb{R}^2 \to S$ defines

a C^1ly varying basis of tangent plane

by $\frac{\partial g}{\partial u_1}, \frac{\partial g}{\partial u_2}$

Note: in previous example $\omega = dx \wedge dy$

defines an orientation on S = upper hemisphere

$g = (r\cos\theta, r\sin\theta, \sqrt{1-r^2})$

$\frac{\partial g}{\partial r} = (\cos\theta, \sin\theta, \frac{-r}{\sqrt{1-r^2}})$

$\frac{\partial g}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$

$$\boxed{\omega\left(\frac{\partial g}{\partial r}, \frac{\partial g}{\partial \theta}\right) = r\cos^2\theta + r\sin^2\theta = r > 0}$$

$h = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

$\frac{\partial h}{\partial \theta} = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$

$\frac{\partial h}{\partial \phi} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$

$\cancel{\omega\left(\frac{\partial h}{\partial \phi}, \frac{\partial h}{\partial \phi}\right) = \sin^2\theta\sin^3\phi - \cos^2\theta}$

for $\phi \in [0, \frac{\pi}{2})$

$$\boxed{\omega\left(\frac{\partial h}{\partial \theta}, \frac{\partial h}{\partial \phi}\right) = -\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi = -\sin\phi\cos\phi}$$

Surface Area

Def If S is an oriented surface its (oriented) area 2-form σ is the 2-form s.t.

$\forall$ $x \in S$ $\qquad \sigma_x(u,v) =$ signed area of parallelogram spanned by u, v

$\forall$ u, v basis for tangent plane to S at x.

e.g. let $S \subset \mathbb{R}^3$ oriented w/ outward normal n

$$\sigma = \cancel{n_1\, dx \wedge d} \quad n_1\, dy \wedge dz + n_2\, dz \wedge dx + n_3\, dx \wedge dy$$

is area 2-form

if u, v are tangent to S at x

i.e. $n \cdot u = n \cdot v = 0$

then

$$\sigma(u,v) = n_1 \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \mp n_2 \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} + n_3 \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$$

$$= \begin{vmatrix} | & | & | \\ n & u & v \\ | & | & | \end{vmatrix} = \text{area of parallelopiped spanned by } n, u, v.$$

since n is unit vector orthogonal to S

$$\begin{vmatrix} | & | & | \\ n & u & v \\ | & | & | \end{vmatrix} = 1 \cdot \left(\text{area of parallelogram spanned by } u, v\right)$$

<u>example</u>: surface of revolution defined by $z = f(r)$

$0 \leq r \leq a$, w/ outward normal pointing in the positive e_3 direction.

$g: (0,a) \times (0, 2\pi) \to S$ $\qquad g(r,\theta) = (r\cos\theta, r\sin\theta, f(r))$

$\frac{\partial g}{\partial r} = (\cos\theta, \sin\theta, f'(r))$ $\qquad \frac{\partial g}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$

$$\left(\frac{\partial g}{\partial r} \times \frac{\partial g}{\partial \theta}\right) \cdot e_3 = r(\cos^2\theta + \sin^2\theta) = r > 0$$

$$n = \frac{\partial g}{\partial r} \times \frac{\partial g}{\partial \theta} \Big/ \left\| \frac{\partial g}{\partial r} \times \frac{\partial g}{\partial \theta} \right\|$$

$$= \frac{1}{\sqrt{1+f'(r)^2}} \left(-\frac{x}{r} f'(r), -\frac{y}{r} f'(r), 1 \right)$$

$$\sigma = \frac{1}{\sqrt{1+f'(r)^2}} \left(-\frac{x}{r} f'(r)\, dy \wedge dz - \frac{y}{r} f'(r)\, dz \wedge dx + dx \wedge dy \right)$$

$\cancel{g^* \sigma =}$

$$g^* (dy \wedge dz) = -x f'(r)\, dr \wedge d\theta$$

$$g^* (dz \wedge dx) = -y f'(r)\, dr \wedge d\theta$$

$$g^* (dx \wedge dy) = r\, dr \wedge d\theta$$

$\cancel{\text{and}}$ $$g^* \sigma = \frac{r f'(r)^2 + r}{\sqrt{1+f'(r)^2}}\, dr \wedge d\theta = r\sqrt{1+f'(r)^2}\, dr \wedge d\theta$$

$$\text{area}(S) = \int_S \sigma = \int_{(0,a)\times(0,2\pi)} g^* \sigma = \int_0^{2\pi} \int_0^a r\sqrt{1+f'(r)^2}\, dr\, d\theta$$