Differential forms (from the beginning)

Multilinear Algebra

Recall $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ space of linear maps $\mathbb{R}^n \to \mathbb{R}$

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n \qquad dx_i(v) = v_i$$

$\{dx_1, \dots, dx_n\}$ are a basis for $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$

given $\phi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ call $a_i = \phi(e_i)$

then $\phi(v) = a_1 dx_1 + \dots + a_n dx_n$

if $\phi = c_1 dx_1 + \dots + c_n dx_n = 0$

then $0 = \phi(e_i) = c_i$ for all $i = 1, \dots, n$

$I = (i_1, \dots, i_k)$ an ordered k-tuple

$dx_I : \mathbb{R}^{nk} \to \mathbb{R}$ by

$$dx_I(v_1, \dots, v_k) = \begin{vmatrix} dx_{i_1}(v_1) & \cdots & dx_{i_1}(v_k) \\ \vdots & & \vdots \\ dx_{i_k}(v_1) & \cdots & dx_{i_k}(v_k) \end{vmatrix}$$

if $v_i = \begin{bmatrix} v_{i,1} \\ \vdots \\ v_{i,k} \end{bmatrix}$

$$dx_I(v_1, \dots, v_k) = \begin{vmatrix} v_{1,1} & \cdots & v_{k,1} \\ \vdots & & \vdots \\ v_{1,k} & \cdots & v_{k,k} \end{vmatrix}$$

i.e. taking rows $i_1, \dots, i_k$ of

matrix $\begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}$

e.g. in $\mathbb{R}^2$ $dx_{(1,2)}(v_1, v_2) = \begin{vmatrix} v_{1,1} & v_{2,1} \\ v_{1,2} & v_{2,2} \end{vmatrix}$

Note dx_I is an <u>alternating</u> multi<u>linear</u> function of $v_1, \dots, v_k$

i.e. linear in each argument and

$$dx_I(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -dx_I(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$

<u>Alternating</u>

Given any k-tuple index I call $I^<$ ~~to~~ be I arranged in increasing order

e.g. $I = \cancel{(3,1,2)}\ (3,2,1) \qquad I^< = (1,2,3)$

$dx_{(3,2,1)} = -dx_{(1,2,3)}$ since I had to do 1-transposition to get from I to $I^<$

In general $\quad dx_{I^<} = (-1)^s dx_I$

$s = \#$ of transpositions

~~Theor~~ Thm The set dx_I, I increasing k-tuple in $\{1, \dots, n\}$ spans the linear space $\Lambda^k[\mathbb{R}^n]^*$ of alternating multi-linear functions from $(\mathbb{R}^n)^k \to \mathbb{R}$.

i.e. $T \in \Lambda^k(\mathbb{R}^n)^*$

$$T = \sum_{I \text{ increasing}}' a_I \, dx_I \qquad a_I = T(e_{i_1}, \dots, e_{i_k})$$

$$\dim \left(\Lambda^k(\mathbb{R}^n)^* \right) = \binom{n}{k}$$

$dx_I(v_1, \dots, v_k)$ is the signed volume of the

projection of parallelogram spanned by $v_1, \dots, v_k$

onto $x_{i_1}, \dots, x_{i_k}$ - plane.

We define the wedge product generality cross product

I, J k, l - tuple respectively

$$dx_I \wedge dx_J := dx_{(I,J)}$$

e.g.

$$dx_{(1,2)} \wedge dx_3 = dx_{(1,2,3)}$$

$$dx_{(1,5)} \wedge dx_{(4,2)} = dx_{(1,5,4,2)} = -dx_{(1,2,4,5)}$$

then we extend by linearity $\omega \in \Lambda^k(\mathbb{R}^n)^*$, $\eta \in \Lambda^l(\mathbb{R}^n)^*$

$$\omega \wedge \eta = \left(\sum' a_I \, dx_I\right)\left(\sum' b_J \, dx_J\right)$$

$$= \sum_{I,J}' a_I b_J \, dx_I \wedge dx_J$$

Ex $\omega = a_1 dx_1 + a_2 dx_2$ $\quad \eta = b_1 dx_1 + b_2 dx_2$

Thm (i) $(\omega + \phi) \wedge \eta = \omega \wedge \eta + \phi \wedge \eta$, $\quad (c\omega) \wedge \eta = c(\omega \wedge \eta)$

(ii) $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$ $\quad \omega \in \Lambda^k$, $\eta \in \Lambda^l$

(iii) $(\omega \wedge \eta) \wedge \phi = \omega \wedge (\eta \wedge \phi)$

Multilinear Algebra (contd...)

last time define $\Lambda^k(\mathbb{R}^n)^*$, wedge / exterior product

$$dx_I \wedge dx_J = dx_{(I,J)}.$$

$\Lambda^k(\mathbb{R}^n)$ is k-fold exterior product of $\mathbb{R}^n$

i.e. define wedge product the same for vectors as we did for dual vectors

$$\begin{cases} v \wedge w = -\cancel{w} w \wedge v \\ v \wedge v = 0 \end{cases}$$ – extend from basis by Bilinearity.

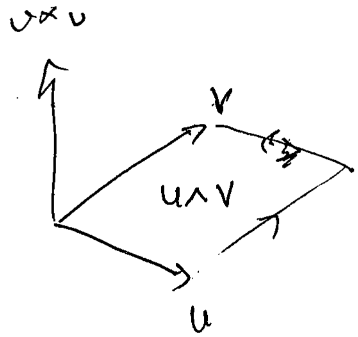
example $u, v \in \mathbb{R}^3$

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3$$
$$v = v_1 e_1 + v_2 e_2 + v_3 e_3$$

$$u \wedge v = (u_1 v_2 - u_2 v_1)\, e_1 \wedge e_2 + (u_1 v_3 - u_3 v_1)\, e_1 \wedge e_3 + (u_2 v_3 - u_3 v_2)\, e_2 \wedge e_3$$

$\{e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_2 \wedge e_3\}$ is a basis for $\Lambda^2(\mathbb{R}^3)$

i.e. $\Lambda^2(\mathbb{R}^3) \cong \mathbb{R}^3$



$u \wedge v$ scales like area

$$(tu \wedge sv) = ts\,(u \wedge v)$$

note that under the identification

$$e_1 \wedge e_2 \mapsto e_3 \qquad e_1 \wedge e_3 \mapsto e_2 \qquad e_2 \wedge e_3 \mapsto e_1$$

wedge and cross product are the same

wedge product generalizes (in a sense) the cross product.

$\Lambda^k(\mathbb{R}^n) = \{$space of k-dim parallelograms in $\mathbb{R}^n\}$

$\Lambda^k(\mathbb{R}^n)^* = \{$ " " linear functionals on $\Lambda^k(\mathbb{R}^n)\}$

Differential forms

$\Lambda^k(\mathbb{R}^n)^*$ is a natural linear space for integration on k-dim surfaces, but we want the corresponding nonlinear space.

0-form is a smooth function on $\mathbb{R}^n$

~~n-form~~ ~~for~~ n-form on $\mathbb{R}^n$ is

$$\omega = f(x)\, dx_1 \wedge \cdots \wedge dx_n \qquad x \in \mathbb{R}^n, \; f \text{ smooth}$$

a differential k-form is

$$\omega = \sum_{\substack{\text{incr. } k\text{-tuple} \\ I}} f_I(x)\, dx_I = \sum_{i_1 < \cdots < i_k} f_I(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$f_I : \mathbb{R}^n \to \mathbb{R} \quad \text{smooth}$$

We can do the natural algebraic operations

- adding 2 k-forms
- scalar mult (mult by a 0-form)
- wedge k, l form.

$A^k(\mathbb{R}^n)$ = vector space of k-forms on $\mathbb{R}^n$

$\omega \in A^k(\mathbb{R}^n)$ $\quad \eta \in A^l(\mathbb{R}^n)$ $\quad \cancel{\phi \in A} \quad \phi \in A^m(\mathbb{R}^n)$

- $k = l = m$, $\quad \omega + (\eta + \phi) = (\omega + \eta) + \phi$
- $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$
- $(\omega \wedge \eta) \wedge \phi = \omega \wedge (\eta \wedge \phi)$
- $k = l$ $\quad (\omega + \eta) \wedge \phi = \omega \wedge \phi + \eta \wedge \phi$

Exterior derivative

f a 0-form then

$df(x) = Df(x)$ is naturally a 1-form

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

if ω is a k-form $\quad \omega = \sum_I f_I(x)\, dx_I$

defn $$d\omega = \sum_I{}' df_I \wedge dx_I = \sum_I{}' \sum_{j=1}^{n}{}' \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$$

~~Examples~~ ~~Note: formal product rule~~

~~Note~~ $\cancel{d(dx_1 \wedge \dots \wedge dx_k) = d^2 x_1}$

Examples

[1] $f : \mathbb{R} \to \mathbb{R}$ smooth

$$df(x) = f'(x)\, dx$$

[2] $\omega = y\,dx + x\,dy \in A^1(\mathbb{R}^2)$

$$d\omega = dy \wedge dx + dx \wedge dy = 0$$

[3] $\omega = -y\,dx + x\,dy$

$$d\omega = -dy \wedge dx + dx \wedge dy = 2\,dx \wedge dy$$

[4] $\omega = P\,dx + Q\,dy$

$$d\omega = \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

Thm $\omega \in A^k(\mathbb{R}^n)$, $\eta \in A^l(\mathbb{R}^n)$, f smooth

- $k = l$ $\quad d(\omega + \eta) = d\omega + d\eta$
- $d(f\omega) = df \wedge \omega + f\,d\omega$
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
- $d(d\omega) = 0$

proof $\quad d(\omega\wedge\eta) = d(fg)\wedge dx_I\wedge dx_J$

take $\quad \omega = f\,dx_I \quad = (g\,df + f\,dg)\wedge dx_I\wedge dx_J$

$\eta = g\,dx_J \quad = g\,df\wedge dx_I\wedge dx_J + f\,dg\wedge dx_I\wedge dx_J$

$= d\omega\wedge\eta + (-1)^k\,\omega\wedge d\eta$

$\omega = f\,dx_I$

$$d(d\omega) = d\left(\sum_{j=1}^{n}{}' \frac{\partial f}{\partial x_j}\,dx_j\wedge dx_I\right)$$

$$= \sum_{i=1}^{n}{}' \sum_{j=1}^{n}{}' \frac{\partial^2 f}{\partial x_i\partial x_j}\,dx_i\wedge dx_j\wedge dx_I$$

$$= \sum_{i<j}{}' \left(\frac{\partial^2 f}{\partial x_i\partial x_j} - \frac{\partial^2 f}{\partial x_j\partial x_i}\right) dx_i\wedge dx_j\wedge dx_I$$

$= 0$ by equality of mixed partials.

$\square$

so $\quad d : A^k(\mathbb{R}^n) \to A^{k+1}(\mathbb{R}^n)$

$d^2 = 0$

We can integrate k-form on $\mathbb{R}^n$ naturally, ~~how~~

or even k-form on k-dim subspace of $\mathbb{R}^n$

but how to define integral of k-dim surface?

ω n-form in $\mathbb{R}^n$ $\Rightarrow$ $\omega = \sum f(x)\, dx_1 \wedge \dots \wedge dx_n$

$R \subset \mathbb{R}^n$, partition into (oriented) R_j

~~take~~ i.e. each R_j parallelogram in $\mathbb{R}^n$

is a point x_j and

$a_1 e_1 \wedge \dots \wedge a_n e_n \in A^n(\mathbb{R}^n)$

$$\int_R \omega = \lim_{\text{mesh}(P) \to 0} \sum_i \underbrace{f(x_j)\, dx_{(1,\dots,n)}(a_1 e_1 \wedge \dots \wedge a_n e_n)}_{= f(x_j)\,|R_j|}$$

Pull backs

(e.g. think of a smooth curve)

$g : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ smooth, $\omega \in A^k(\mathbb{R}^n)$

we define the pullback $g^* \omega$ as follows if $\omega = f$ a 0-form

$$g^* f = f \circ g$$

(i.e. like evaluating f along param curve)

to pull back the basis 1-forms

~~g*~~ calling $x = g(u)$

$$g^* dx_i = dg_i = \sum_{j=1}^{m}{}' \frac{\partial g_i}{\partial u_j} du_j$$

$$g^* dx_I = g^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = dg_{i_1} \wedge \cdots \wedge dg_{i_k} = dg_I$$

and

$$g^*\left(\sum_I{}' f_I \, dx_I\right) = \sum_I{}' (f_I \circ g) \, dg_I$$

Examples $\boxed{1}$ $g: \mathbb{R} \to \mathbb{R}^2$ $g(t) = (\cos t, \sin t)$

$$g^* dx = -\sin t \, dt \qquad g^* dy = \cos t \, dt$$

if $\omega = x\,dx + y\,dy$

$$g^*\omega = \cos t\,(-\sin t)\,dt + \sin t \cos t \, dt = 0$$

if $\quad \omega = -y\,dx + x\,dy$

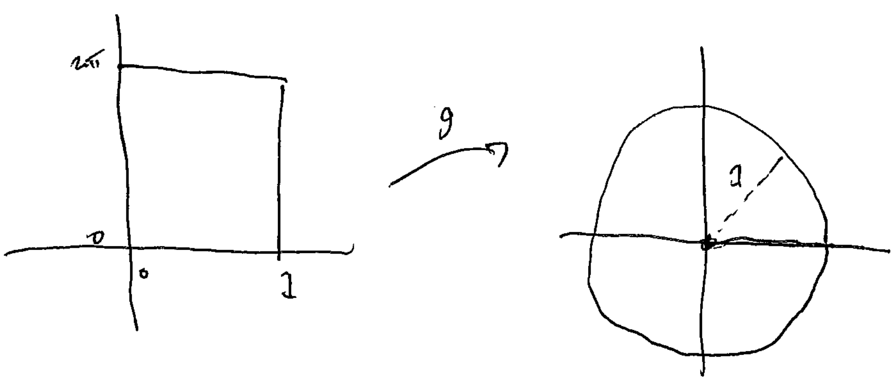
$$g^*\omega = (-\sin t)(-\sin t\,dt) + \cos t\,(\cos t\,dt)$$

$$= (\sin^2 t + \cos^2 t)\,dt = dt$$

$\boxed{2}$ $\quad g : \mathbb{R}^2 \to \mathbb{R}^2$

~~$g(s,t) = (s\cos t, s$~~

$$g(r,\theta) = (r\cos\theta, r\sin\theta)$$



$\omega = x\,dx + y\,dy$

$$g^*\omega = r\cos\theta\, g^*dx + r\sin\theta\, g^*dy$$

$$= r\cos\theta\,(\cos\theta\,dr - r\sin\theta\,d\theta) + r\sin\theta\,(\sin\theta\,dr + r\cos\theta\,d\theta)$$

$$= r(\cos^2\theta + \sin^2\theta)\,d\theta + 0\,d\theta = r\,dr$$

$\omega = dx \wedge dy$

$$g^*\omega = \cancel{\#}\, g^*dx \wedge g^*dy = (\cos\theta\,dr - r\sin\theta\,d\theta) \wedge (\sin\theta\,dr + r\cos\theta\,d\theta)$$

$$= r(\cos^2\theta + \sin^2\theta)\,dr \wedge d\theta = r\,dr \wedge d\theta$$

(i.e. we recover change of variables thm for polar coordinates)

Note the appearance of determinant in the example

$$g^*(dx_I) = dg_{i_1} \wedge \cdots \wedge dg_{i_k} = \left(\sum_{j=1}^{n}{}' \frac{\partial g_{i_1}}{\partial u_j} du_j\right) \wedge \cdots \wedge \left(\sum_{j=1}^{n}{}' \frac{\partial g_{i_k}}{\partial u_j} du_j\right)$$

$$= \sum_{J \text{ incr } k\text{-tuple}}{}' \left(\sum_{\substack{\sigma \\ \text{perm of } J}}{}' \frac{\partial g_{i_1}}{\partial u_{\sigma(j_1)}} \cdots \frac{\partial g_{i_k}}{\partial u_{\sigma(j_k)}} \operatorname{sgn}(\sigma) \right) du_J$$

$$= \sum_{J \text{ incr } k\text{-tuple}}{}' \det\left[\frac{\partial g_I}{\partial u_J}\right] du_J$$

$$= \sum_{J \text{ incr } k\text{-tuple}}{}' \begin{vmatrix} \frac{\partial g_{i_1}}{\partial u_{j_1}} & \cdots & \frac{\partial g_{i_1}}{\partial u_{j_k}} \\ \vdots & & \\ \frac{\partial g_{i_k}}{\partial u_{j_1}} & \cdots & \frac{\partial g_{i_k}}{\partial u_{j_k}} \end{vmatrix} du_{j_1} \wedge \cdots \wedge du_{j_k}$$

Prop: $U \subseteq \mathbb{R}^m$ open $g : U \to \mathbb{R}^n$ smooth, $\omega \in \Omega^k(\mathbb{R}^n)$ then

$$g^*(d\omega) = d(g^*\omega)$$

proof for $k=0$ (chain rule

$$d(g^*f) = d(f \circ g) = \sum_j{}' \left(\sum_i{}' \frac{\partial f}{\partial x_i} \circ g \, \frac{\partial g_i}{\partial u_j} \right) du_j = \sum_i{}' \frac{\partial f}{\partial x_i} \circ g \left(\sum_j{}' \frac{\partial g_i}{\partial u_j} du_j \right)$$

$$= \sum_{i=1}^{n} g^* \frac{\partial f}{\partial x_i} \, g^* dx_i$$

$$= g^* \left(\sum_i \frac{\partial f}{\partial x_i} dx_i \right) = g^* (df)$$

in general

$$g^* (d (f \, dx_I)) = g^* (df \wedge dx_I) = g^* (df) \wedge g^* (dx_I)$$

$$= g^* (df) \wedge dg_I$$

$$= d(g^* f) \wedge dg_I$$

$$= d (g^* f \wedge dg_I) = d (g^* (f \wedge dx_I))$$

$d^2 g_I = 0$

$\square$

Connection w/ integration

n-form $\qquad \omega = f(x) \, dx_1 \wedge \cdots \wedge dx_n$

define $\qquad \displaystyle\int_U \omega = \underbrace{\int_U f}_{\text{old Riemann integral for fns}}$ $\qquad U \subset \mathbb{R}^n$ open, w/ measure 0.

Thm $V \subseteq \mathbb{R}^n$ region, $g: V \to \mathbb{R}^n$ smooth 1-1 w/

$\det(Dg) > 0$. Then for any n-form $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$

on $g(V)$ we ~~have~~ have

$$\int_{g(V)} \omega = \int_V g^* \omega$$

proof: already saw

$$g^*\omega = g^*(f\, dx_1 \wedge \cdots \wedge dx_n)$$

$$= (f \circ g)\, g^*(dx_1 \wedge \cdots \wedge dx_n)$$

$$= f\circ g \det[Dg] = f \circ g\, |\det Dg|$$

(Same as change of vars thm) $\square$

~~Def~~ **Def**: Let $U \subseteq \mathbb{R}^k$, $g: U \to \mathbb{R}^n$ smooth 1-1

w/ Dg has rank k $\forall\, u \in U$.

(c(i)) $M = g(V)$ this is a

parametrized k-dim manifold we define

$$\int_M \omega = \int_V g^* \omega$$

Note: this agrees w/ defn for curves that

we already gave

If g_1, g_2 both param M and

$\det\left[D g_2^{-1} \circ g_1\right] > 0$ (same orientation) then

$$\int_{U_2} g_2^* \omega = \int_{U_1} (g_2^{-1} \circ g_1)^* (g_2^* \omega)$$

$$= \int_{V_1} (g_2 \circ g_2^{-1} \circ g_1)^* \omega$$ ← exercise

$$= \int_{V_1} g_1^* \omega$$