Series of non-negative terms

The geometric series

Thm if $0 \le x < 1$ then

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$$

if $x \ge 1$ then the sum diverges.

proof: If $x \ne 1$ then

$$s_n = 1 + x + \cdots + x^n$$

~~$xs_n = s_{n+1} - 1$~~

$$s_n + x^{n+1} = s_{n+1} = 1 + x s_n$$

so $s_n = \dfrac{1 - x^{n+1}}{1-x}$

$x^{n+1} \to 0$ as $n \to \infty$ for $0 \le x < 1$

if $x = 1$ then $\sum_{n=1}^{n} x^n = n$ diverges.

Thm If $a_1 \ge a_2 \ge \cdots \ge 0$ then $\sum_{n=1}^{\infty} a_n$ converges iff

$$\sum_{k=0}^{\infty} 2^k a_k = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges

Proof: Since the partial sums are monotone we just need to show they are bdd.

$$s_n = a_1 + \cdots + a_n$$

$$t_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k}$$

for $n < 2^k$

$$s_n \le a_1 + (a_2 + a_3) + \cdots + (\underbrace{a_{2^k} + \cdots + a_{2^{k+1}-1}}_{\substack{2^{k+1}-2^k = 2^k \text{ terms} \\ \text{all } \le a_{2^k}}}) \text{ terms}$$

$$\le a_1 + 2a_2 + \cdots + 2^k a_{2^k} = t_k$$

so if t_k bdd then s_n bdd.

If $n > 2^k$ then

$$s_n \geq \cancel{a_1 + (a_2 + a} \; a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k})$$

$$\geq \cancel{2a_2 +} \; \frac{1}{2} a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k}$$

$$= \frac{1}{2} t_k$$

so if s_n bdd then t_k bdd.

Thm $\sum' \frac{1}{n^p}$ converges ~~iff~~ if $p > 1$, diverges if ~~p~~ $p \leq 1$

proof: if $p \leq 0$ terms don't $\to 0$.

if $p > 0$ terms are monotone decr so

$\sum' \frac{1}{n^p}$ conv iff $\sum_{k=0}^{\infty}{}' \frac{1}{2^{kp}} 2^k$ converges

this is geometric w/ $2^{(1-p)}$ as the parameter. if $p > 1$
$2^{1-p} < 1$ if $p \leq 1$ then $2^{(1-p)} \geq 1$

$\square$

Thm If $p > 1$

$$\sum_{n=2}^{\infty}{}' \frac{1}{n(\log n)^p}$$ converges, otherwise it diverges

log is always log base e, log is monoton incr.

proof: we can just study

$$\sum_{k=1}^{\infty}{}' \frac{1}{2^k (\log 2^k)^p} 2^k = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty}{}' \frac{1}{k^p}$$ converges if $p > 1$
diverges if $p \leq 1$

Root and Ratio Tests (useful for things that are "like" geometric series)

Thm (Root test) Given $\sum_1 a_n$ set $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ then

(a) if $\alpha < 1$ sum converges

(b) if $\alpha > 1$ sum diverges

(c) if $\alpha = 1$ no info

proof: let $1 > \beta > \alpha$ since α is limsup

$\exists$ N s.t. $n \geq N \Rightarrow (|a_n|)^{1/n} \leq \beta$

then $|a_n| \leq \beta^n$ $\forall$ $n \geq N$

$\sum_{n\geq 1}^{\infty} \beta^n$ converges $\Rightarrow$ $\sum_1 a_n$ converges

by the comparison thm.

(b) let ~~$1 < \beta < \alpha$~~ ~~then $\exists$ a sequence~~

~~$n_k \to \infty$ s.t.~~

~~and let $1 < \beta < \alpha$ $\exists$ n s.t. $\beta^n > M$~~

let $1 < \beta < \alpha$ then $\exists$ a sequence $n_k \to \infty$ s.t.

$|a_{n_k}| \geq \beta^{n_k} \geq 1$ so $a_{n_k} \not\to 0$

$\Rightarrow$ $a_n \not\to 0$ $\Rightarrow$ $\sum_1 a_n$ does not converge

(c) for example $a_n = \frac{1}{n}$, $b_n = \frac{1}{n^2}$

$\sum a_n = +\infty$ $\sum b_n$ converges

but $\lim_{n\to\infty} \frac{1}{n^{1/n}} = \lim_{n\to\infty} \frac{1}{n^{2/n}} = 1$.

i.e. root test helps for series w/ exponential behavior but not polynomial

<u>Thm</u> (Ratio test) the series $\sum a_n$

(i) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

(ii) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ (nonzero terms) for all $n \geq n_0$, some fixed n_0.

<u>proof</u> let $\overline{\lim} \left|\frac{a_{n+1}}{a_n}\right| < r < 1$

$\exists$ N s.t. $n \geq N \Rightarrow |a_{n+1}| \leq r|a_n|$

so $|a_n| \leq r^{n-N} |a_N|$

so since $\sum r^n$ converges, by comp thm

a_n converges.

(iii) if $|a_{n+1}| \geq |a_n|$ $\forall$ $n \geq n_0$

$\Rightarrow$ $a_n \not\to 0$

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again ratio test is about series w/ exponential type growth / decay.

Examples see HW or book.

~~Absolute Convergence~~

Summation by Parts (like integration by parts)

Thm Given two sequences a_n, b_n put

$$A_n = \sum_{k=0}^{n} a_k \quad \text{if } n \geq 0, \quad A_{-1} = 0$$

if $0 \leq p \leq q$ then

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

proof: $$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$

$$= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + \left(A_q b_q - A_{p-1} b_p\right)$$

Thm Suppose

(i) the partial sums A_n of $\sum a_n$ are bdd

(ii) $b_0 \geq b_1 \geq \cdots$

(iii) $\lim_{n\to\infty} b_n = 0$

Then $\sum a_n b_n$ converges

proof: let M s.t. $|A_n| \leq M$ $\forall n$.

let $\varepsilon > 0$ and N suff large s.t. $b_n \leq \varepsilon$

and $q \geq p \geq N$

$$\left| \sum_{n=p}^{q} a_n b_n \right| = \left| \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right|$$

all positive terms ~~except for last one~~

$$\leq M\left(\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p \right) \cancel{\leq M\left(\left| \sum_{n=p}^{q-1} (b_n - b_{n+1}) \right| + |b_q| + |b_p| \right)}$$

$$\leq 2M b_p \leq 2M\varepsilon \cancel{\leq M\left(|b_q - b_p| + |b_q| + |b_p| \right)}$$

$$\cancel{\leq 4M\varepsilon}.$$

Thm (Alternating Series)

Suppose (i) $|c_1| \geq |c_2| \geq \cdots$

(ii) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$

proof $a_n = (-1)^{n+1}$, $b_n = |c_n|$

$$\sum_{n=1}^{N} (-1)^{n+1} = \begin{cases} +1 & N \text{ odd} \\ 0 & N \text{ even} \end{cases}$$ bounded clearly.

Absolute Convergence

the series $\sum_1^\infty a_n$ is said to converge absolutely if

$\sum_1^\infty |a_n|$ converges

Thm If $\sum_1^\infty a_n$ converges absolutely then $\sum_1^\infty a_n$ converges

proof: $\left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k|$

e.g. $\sum_1^\infty \frac{(-1)^n}{n}$ ~~does~~ converges but not absolutely.

Absolutely convergent series can usually be treated like finite sums (can ~~add and~~ multiply and rearrange terms)

this is not the case for non-absolutely convergent sums which must be dealt with very carefully.

Power Series

given a sequence c_n of complex numbers

$$\sum_{n=0}^{\infty} c_n z^n \quad \text{is called } \underline{\text{a power series}}$$

c_n are called the $\underline{\text{coefficients}}$

In general there is a disk

$B(0, R_*)$ s.t. the series converges

for $|z| < R_*$, diverges for $|z| > R_*$

and has some complicated behavior

on the circle $|z| = r_*$.

$\underline{\text{Thm}}$ Given $\sum c_n z^n$ put

$$\alpha = \limsup_{n \to \infty} |c_n|^{1/n} \quad , \quad R = \frac{1}{\alpha}$$

(If $\alpha = 0$, $R = +\infty$, if $\alpha = +\infty$, $R = 0$)

Then $\sum c_n z^n$ converges if $|z| < R$ diverges

if $|z| > R$.

proof put $a_n = c_n z^n$ and apply root test

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = |z| \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = |z|\alpha$$

so if $|z|\alpha < 1 \Rightarrow$ convergence

$|z|\alpha > 1 \Rightarrow$ divergence $\square$

examples

- $\sum n^n z^n$ has $R=0$
- $\sum \frac{z^n}{n!}$ has $R=+\infty$

(ratio test)

- $\sum z^n$ has $R=1$
- $\sum_1 \frac{z^n}{n}$ has $R=1$

converges for $|z|=1$ w/ $z \neq 1$

- $\sum_1 \frac{z^n}{n^2}$ has $R=1$

converges for all $|z|=1$

since $\sum \frac{1}{n^2}$ converges.

Addition and Multiplication of Series

Thm If $\sum a_n = A$ and $\sum b_n = B$ then

$$\sum (a_n + b_n) = A + B \quad \text{and}$$

$$\sum c\, a_n = cA \quad \text{for any } c \in \mathbb{R}$$

proof Call the partial sums

$$A_k = \sum_{n=1}^{k} a_n \qquad B_k = \sum_{n=1}^{k} b_n$$

$$A_k + B_k = \sum_{n=1}^{k} (a_n + b_n)$$

$$\longrightarrow A + B \quad \text{as } k \to \infty$$

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product of series

given $\sum a_n$, $\sum b_n$ define

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Motivated by

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2$$

$$+ \cdots$$

Thm Suppose

(i) $\sum_{n=0}^{\infty} a_n$ converges absolutely

(ii) $\sum_{n=0}^{\infty} a_n = A$

(iii) $\sum_{n=0}^{\infty} b_n = B$

(iv) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

then $$\sum_{n=0}^{\infty} c_n = AB .$$

proof: see Book p. 74 Thm 3.50.

Thm If $\sum a_n$, $\sum b_n$, $\sum c_n$ converge to A, B, C then $C = AB$.

Rearrangements

let k_n be an enumeration of $\mathbb{N}_+$

(i.e. $n \mapsto k_n$ is 1-1 and onto map)

put $a_n' = a_{k_n}$ $(n = 1, 2, 3, \ldots)$

we say $\Sigma_1^\infty a_n'$ is a <u>rearrangement</u>

of $\Sigma_1^\infty a_n$.

if s_n and s_n' are the partial sums

of $\Sigma_1^\infty a_n$ and $\Sigma_1^\infty a_n'$ resp

they can consist of totally different terms

(although eventually each term needs to appear in both)

for example !!

$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ which converges

$s < 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

and $s' = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6}$

two positive terms followed by a negative term.

$$\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > \frac{1}{2k} - \frac{1}{2k} = 0$$

so $$s_3' < s_6' < s_9' < \cdots$$

so $$\limsup_{n\to\infty} s_n' > s_3' = \frac{5}{6}.$$

<u>Thm</u> Let $\sum a_n$ be a series of real numbers converging but not absolutely. Suppose

$$-\infty \le \alpha \le \beta \le \infty$$ then $\exists$ a rearrangement

$\sum a_n'$ w/ partial sums s_n' s.t.

$$\liminf s_n' = \alpha \qquad \limsup s_n' = \beta.$$

Proof:

let $p_n = \max\{a_n, 0\}$ $q_n = \max\{-a_n, 0\}$

then $p_n + q_n = |a_n|$

so $\Sigma_1^\infty p_n$ and $\Sigma_1^\infty q_n$ must

diverge otherwise

(if both converge) $\Sigma_1^\infty (p_n + q_n) = \Sigma_1^\infty |a_n|$ would converge.

if one converges and the other diverges.

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} (p_n - q_n) = \sum_{n=1}^{N} p_n - \sum_{n=1}^{N} q_n$$

then $\Sigma_1^\infty a_n$ diverges, again a contradiction

Call P_n the non-negative terms of $\Sigma_1^\infty a_n$

in the order they occur

and Q_n the absolute value of the negative

terms in the order they occur

$\Sigma_1^\infty P_n$ and $\Sigma_1^\infty Q_n$ also diverge

since they are the same (up to terms that

are zero) as p_n, q_n resp.

we will construct m_n, k_n s.t.

$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots P_{m_2} - Q_{k_1+1} - \cdots$$

which is a rearrangement of given satisfies the desired property.

let $\beta_n, \alpha_n \in \mathbb{R}$ $\quad \alpha_n \to \alpha$, $\beta_n \to \beta$

$\alpha_n < \beta_n$ and $\beta_1 > 0$.

let m_1, k_1 smallest s.t.

$$P_1 + \cdots + P_{m_1} > \beta_1$$

$$P_1 + \cdots + P_{m_1} - Q_1 + \cdots - Q_{k_1} < \alpha_1$$

let m_2, k_2 s.t.

$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} > \beta_2$$

$$\left(\qquad \qquad \qquad \right) - Q_{k_1+1} - \cdots - Q_{k_2} < \alpha_2$$

and continue in this way.

this is possible since $\sum_{n\geq N}' P_n$ and $\sum_{n\geq N}' Q_n = +\infty$

$\forall \quad N \in \mathbb{N}_+$.

Call x_n, y_n resp the partial sums

ending in P_{m_n} and Q_{k_n}

then $$|x_n - \beta_n| \leq P_{m_n}$$

$$|y_n - \alpha_n| \leq Q_{k_n}$$

Since $\sum_1 a_n$ conv $\Rightarrow$ P_{m_n}, $Q_{k_n} \to 0$ as $n \to \infty$

so $x_n \to \beta$, $y_n \to \alpha$ $\square$

<u>Thm</u> Let $\sum' a_n'$ be a rearrangement of an absolutely convergent series $\sum' a_n$. Then $\sum_1 a_n'$ converges and

$$\sum_1 a_n' = \sum_1' a_n = A$$

<u>proof</u>: Let $A_N' = \sum_{n=1}^{N}{}' a_n'$ and $A_N = \sum_{n=1}^{N}{}' a_n$

then $$|A_N' - A| \leq |A_N' - A_M| + |A_M - A|$$

let N_1 s.t. $|A_{M} - A| \leq \varepsilon$ for $M \geq N_1$.

let N_2 s.t. $\forall$ ~~$N \geq$~~ $n, m \geq N_2$

$$\left|\sum_{k=n}^{m} a_k\right| \leq \varepsilon \quad \text{(i.e. } |A_m - A_n| \leq \varepsilon\text{.)}$$

fix $M \geq M_1 \vee M_2$

now since $a_n' = a_{k_n}$ is a rearrangement

~~$\exists K$ s.t. for~~

$\exists$ $n_1, \ldots, n_M$ s.t. $k_{n_\ell} = \ell$

let $N \geq \max\{n_1, \ldots, n_M\}$

then
$$|A_N' - A_M| = \left|\sum_{n=1}^{N} a_{k_n} - \sum_{n=1}^{M} a_n\right|$$

$$= \left|\sum_{\substack{n \in \{1,\ldots,N\} \\ k_n \geq M}} a_{k_n}\right| \leq \left|\sum_{M+1}^{\infty} a_n\right| \leq \varepsilon$$

so
$$|A_N' - A| \leq |A_N' - A_M| + |A_M - A|$$

$$\leq 2\varepsilon$$

$\square$