Sequences and Series

A sequence $p_n \in X$ metric space is said to <u>converge</u> if

$\exists\ p$ s.t. $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$

s.t. $n \geq N \Rightarrow d(p_n, p) \leq \varepsilon$.

We write $p_n \to p$ or $\lim_{n \to \infty} p_n = p$.

Note that convergence depends on the metric d and the space X

e.g. $\{\frac{1}{n} : n \in \mathbb{N}\}$ $p_n = \frac{1}{n}$

converges in $(\mathbb{R}, |x-y|)$

but not in $((0,\infty), |x-y|)$

and not in $(\mathbb{R}, d_0)$

discrete metric.

Remember can view a sequence as a map
$p_n : \mathbb{N}_+ \to X$

p_n called <u>bounded</u> if range of p_n in X is bdd.

<u>Thm</u> Let p_n sequence in (X, d)

(a) $p_n \to p$ iff for every $r>0$ $p_n \in B(p, r)$ except for finitely many n.

(b) $p, p' \in X$ and $p_n \to p$ and $p_n \to p'$ then $p = p'$

(c) if p_n converges then p_n bounded

(d) If $E \subset X$ and p is a limit point of E then $\exists$ a sequence $p_n \in E$, $p_n \to p$.

<u>proof</u>: (c): p_n converges to some $p \in X$ by assumption

so $\exists$ N s.t. $\forall$ $n \geq N$ $p_n \in B(X, 1)$

let $M = \max\limits_{1 \leq n < N} d(p, p_n) < +\infty$ since max is finite.

then $p_n \in B(p, M+1)$ for all n

(d): p limit pt $\Rightarrow$ for each n $\exists$ $p_n \in B(p, \frac{1}{n})$

sequence $p_n \to p$.

~~Connected Sets~~

~~Thm (Baire Category in $\mathbb{R}^n$)~~

In $\mathbb{R}^n$ we want to understand the interaction of limits and algebraic operations

Thm Suppose $s_n, t_n \in \mathbb{R}$ and $\lim s_n = s$, $\lim t_n = t$. Then:

(i) $\lim (s_n + t_n) = s+t$

(ii) $\lim\limits_{n\to\infty} c s_n = cs$, $\lim\limits_{n\to\infty} (c+s_n) = s_n$

$\forall\ c \in \mathbb{R}$

(iii) $\lim\limits_{n\to\infty} s_n t_n = st$

(iv) $\lim\limits_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ if $s_n \neq 0$ $\forall\ n$ and $s \neq 0$.

proof: (i) Let $\varepsilon > 0$ $\exists\ N_1, N_2$ s.t.

$$n \geq N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$$

$$n \geq N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2}$$

Call $N = \max\{N_1, N_2\}$ so that $n \geq N$

$$\Rightarrow \left|s_n + t_n - (s+t)\right| \leq |s_n - s| + |t_n - t| \leq \varepsilon.$$

(ii) trivial

(iii) some setup

$$|s_n t_n - st| = |s_n(t_n - t) + t(s_n - s)|$$

$$\le |s_n||t_n - t| + |t||s_n - s|$$

Since convergent sequences are bdd

$\exists$ M_1, M_2 s.t. $|s_n| \le M_1$, $|t_n| \le M_2$ $\forall n$

let N_1, N_2 s.t. $n \ge N_1$

$$\Rightarrow |s_n - s| \le \frac{\varepsilon}{2M_2}$$

$$n \ge N_2 \Rightarrow |t_n - t| \le \frac{\varepsilon}{2M_1}$$

Then $n \ge N_1 \vee N_2 \Rightarrow |s_n t_n - st| \le \varepsilon$

(iv) let n ~~suff~~ large s.t. ~~$|s_n - s|$~~ $|s_n| > \frac{1}{2}|s|$

$\sim$

Thm: Suppose $x_n \in \mathbb{R}^d$ $n \in \mathbb{N}_+$

(i) Then $x_n \to x$ (Euclidean metric)

iff $x_n^i \to x^i$ $\forall$ $1 \le i \le d$.

(ii) $x_n, y_n \in \mathbb{R}^d$, $\beta_n \in \mathbb{R}$

$x_n \to x$, $y_n \to y$, $\beta_n \to \beta$ as $n \to \infty$

then $x_n + y_n \to x + y$, $x_n \cdot y_n \to x \cdot y$

and $\beta_n x_n \to \beta x$ as $n \to \infty$

proof: (i) $|x_n^i - x^i| \le \|x_n - x\|$

is clear from defn. Then it is easy.

(ii)

$x_n \cdot y_n = x_n^1 y_n^1 + \cdots + x_n^d y_n^d$

Use part (i) to get $x_n^i, y_n^i \to x^i, y^i$ resp and use that convergence $\Rightarrow$ convergence of product.

Subsequences

given a sequence p_n, consider a sequence n_k of positive integers s.t. $n_1 < n_2 < \cdots$ then the sequence p_{n_i} is called a <u>subsequence</u> of p_n

if p_{n_i} converges then the limit is called a <u>subsequential limit</u> of p_n

$p_n \to p$ iff every subsequence of p_n converges to p.

<u>Thm</u> (a) If p_n is a sequence in (X,d) cpct metric space then p_n has a convergent subsequence

(b) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

<u>proof</u>: Call E = range of p_n $n \in \mathbb{N}_+$

if E finite then $\exists$ subsequence

$n_1 < n_2 < \cdots$ s.t. $p_{n_k} = p_0 \in E$

$p_{n_k} \to p$ trivially

If E infinite then by X cpct E has

a limit pt. $q \in X$.

~~for each $k \in \mathbb{N}_+$ $\exists$~~

let n_1 s.t. $d(p_{n_1}, q) < 1$

given n_k $\exists$ $n_{k+1} > n_k$

s.t. $d(p_{n_k}, q) < \frac{1}{k}$

(otherwise $E \cap B(q, \frac{1}{k})$ is finite)

then $p_{n_k} \to q$.

(6) every bounded subset of $\mathbb{R}$

is a subset of a closed and bdd

(hence cpct) interval.

Thm The subsequential limits of $\{p_n\}$ form a closed subset of E.

proof: Call $E^* =$ set of subsequential limits of $\{p_n\}$

~~suppose $q \in (E^*)'$~~

let $q \in (E^*)'$

let $n_1 = 1$

given n_k define n_{k+1} as follows

since $q \in (E^*)'$ $\exists$ $x \in E^* \cap B(q, \frac{1}{k})$

since x is a subsequential limit

Cauchy Sequences

A sequence p_n in a metric space (X, d) is said to be a Cauchy sequence if

$\forall \; \varepsilon > 0 \quad \exists N$ s.t.

$$m, n \geq N \Rightarrow d(p_n, p_m) \leq \varepsilon.$$

Def: Let $E \neq \emptyset$ subset of metric space X.

The diameter of E is the supremum of $d(p, q)$ over $p, q \in E$.

A sequence is Cauchy iff

$$E_N = \{p_N, p_{N+1}, \ldots\}$$

satisfies $\operatorname{diam} E_N \to 0$ as $N \to \infty$.

Thm (a) $\operatorname{diam} \overline{E} = \operatorname{diam} E$

(b) if K_n is a nested sequence of cpct sets s.t. $\lim_{n \to \infty} \operatorname{diam} K_n = 0$

then $\bigcap_{n=1}^{\infty} K_n$ is a singleton proof: since $E \subset \overline{E}$

$$\operatorname{diam} E \leq \operatorname{diam} \overline{E}$$

$$\sup\{d(p,q) : p,q \in E\} \qquad \sup\{d(p,q) : p,q \in \overline{E}\}$$

let $p, q \in \overline{E}$ and $\varepsilon > 0$

$\exists$ $p', q' \in E$ s.t. $d(p',p) < \varepsilon$
$d(q',q) < \varepsilon$

so $d(p,q) \leq d(p,p') + d(q',p') + d(q',q)$

$\leq 2\varepsilon + d(q',p')$

$\leq 2\varepsilon + \operatorname{diam} E$

so (since $2\varepsilon + \operatorname{diam} E$ is an UB for $\{d(p,q) : p,q \in \overline{E}\}$)

$$\operatorname{diam} \overline{E} \leq 2\varepsilon + \operatorname{diam} E$$

since ε was arbitrary

$$\operatorname{diam} \overline{E} \leq \operatorname{diam} E \qquad \Omega$$

Thm (i) in any metric space convergent sequences are Cauchy

(ii) If X is compact and p_n is Cauchy then p_n converges to a point of X

(iii) in $\mathbb{R}^d$ every Cauchy sequence converges

proof: Let p_n cauchy

call $E_N = \{p_N, p_{N+1}, \dots\}$

then $\lim_{N \to \infty} \operatorname{diam}(\overline{E_N}) = 0$

since $\overline{E_N}$ closed $\subset X$ cpt

$\Rightarrow$ $\overline{E_N}$ compact, nested, nonempty

so $\exists!$ $p \in X$ s.t. $p \in \bigcap_{N=1}^{\infty} \overline{E_N}$.

let $\varepsilon > 0$ $\exists$ $N_0 > 0$ st. $\operatorname{diam} \overline{E_{N_0}} < \varepsilon$

then for $n \geq N_0$

$p \in \overline{E_{N_0}}$ $p_n \in \overline{E_{N_0}}$ so $d(p_n, p) \leq \operatorname{diam} \overline{E_{N_0}} < \varepsilon$ $\square$

(iii) in $X = \mathbb{R}^d$ closed and bdd sets are cpct. Lets show that ~~#~~ the range of p_n is ~~corr~~ bdd so $\overline{E_N}$ are cpct.

let N_0 s.t. $\operatorname{diam} \overline{E_{N_0}} < 1$

then $\{p_1, \dots\} = \{p_1, \dots, p_{N_0 - 1}\} \cup E_{N_0}$

so $d(p_n, p_{N_0}) < \max\{1, d(p_1, p_{N_0}), \dots, d(p_{N_0-1}, p_{N_0})\}$

$\square$

<u>Def</u> A metric space is called <u>complete</u> if every Cauchy sequence converges.

Every closed subset of a complete space is complete.

e.g. $(\mathbb{Q}, |x-y|)$ is not complete.

A sequence in $\mathbb{R}$ is called <u>monotone</u> (increasing/decr)

if $\quad s_{n+1} \geq s_n \quad n = 1, 2, \ldots$

$$\left(\text{or} \quad s_{n+1} \leq s_n \quad n = 1, 2, \ldots \right)$$

<u>Thm</u> If s_n is a monotone sequence in $\mathbb{R}$ then s_n converges iff it is bdd

<u>proof</u> Suppose s_n monotone increasing

if s_n bdd call $s = \sup_{n=1,2,\ldots} s_n$

then $\quad s_n \leq s \quad \forall \; n \in \mathbb{N}_+$

and $\quad \forall \; \varepsilon > 0 \quad \exists \; N \quad$ s.t.

$$s_N > s - \varepsilon$$

by monotonicity $\quad \forall \; n \geq N$

$$s \geq s_n \geq s_N > s - \varepsilon$$

so $\quad |s_n - s| < \varepsilon$.

We can define convergence in $\mathbb{R}_*$ so that all monotone sequences converge.

say $s_n \to +\infty$

if $\forall$ $M \in \mathbb{R}$ $\exists N \in \mathbb{N}_0$

s.t. $n \geq N \Rightarrow s_n \geq M$.

Say $s_n \to -\infty$

similar defn

let s_n be a sequence in $\mathbb{R}$

call $E = \{$ subsequential limits of $s_n \}$ in $\mathbb{R}_*$

Call $s^* = \sup E$

$s_* = \inf E$

these numbers are called the upper and lower limits of s_n

$$s^* = \limsup_{n \to \infty} s_n = \overline{\lim_{n\to\infty}} s_n$$

$$s_* = \liminf_{n \to \infty} s_n = \underline{\lim_{n\to\infty}} s_n$$

Thm let s_n a sequence in $\mathbb{R}$, ~~E, s^* as above then~~

(a) ~~$\exists$~~ $\exists$ a subsequence s_{n_k} s.t.

$$s_{n_k} \to s^* \quad \in$$

(b) if $x > s^*$ $\exists$ N s.t. $\forall n \geq N$

$$s_n < x \quad .$$

(c) $$\limsup_{n \to \infty} s_n = \lim_{N \to \infty} \sup_{n \geq N} s_n$$

$$\liminf_{n \to \infty} s_n = \lim_{N \to \infty} \inf_{n \geq N} s_n$$

Proof let's prove (c), note $\sup_{n \geq N} s_n$ monotone decr hence convergent

let $s \in E$ then $\exists$ $s_{n_k} \to s$

$\forall$ $\varepsilon > 0$ $\exists$ K s.t. $\forall k \geq K$

$$\cancel{s_{n_k} \leq s + \varepsilon}$$

$$|s_{n_k} - s| \leq \varepsilon$$

$$s \leq s_{n_k} + \varepsilon \leq \sup_{n \geq n_k} s_n + \varepsilon$$

sending $k \to \infty$

$$s \leq \lim_{N \to \infty} \sup_{n \geq N} s_n + \varepsilon$$

so $\limsup_{n\to\infty} s_n \le \lim_{N\to\infty} \sup_{n\ge N} s_n$

on the other hand lets show that

$\lim_{N\to\infty} \sup_{n\ge N} s_n \in E$ so we get

the other direction and part (a)

at the same time

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given $k \in \mathbb{N}_+$

since $\lim_{N\to\infty} \sup_{n\ge N} s_n \le \sup_{n\ge n_{k-1}} s_n$ for any M

$\exists\ n_k > n_{k-1}$ s.t. $s_{n_k} \ge$

Series

for this we need an addition operation so we restrict to , $\mathbb{R}$, $\mathbb{C}$ ($\mathbb{R}^d$ also possible)

Sums $\sum_{n=p}^{q} a_n = a_p + \cdots + a_q$

partial sums $s_n = \sum_{k=1}^{n} a_k$

if s_n converges w/ limit s

we define the infinite sum

$$s = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n .$$

s_n converges iff it is Cauchy, which can be restated as

Thm $\sum_{n=1}^{\infty} a_n$ converges iff $\forall \; \varepsilon > 0 \; \exists \; N$ s.t.

$$\left| \sum_{k=n}^{m} a_n \right| \leq \varepsilon \quad \text{for any } m \geq n \geq N.$$

Thm $\sum' a_n$ converges $\Rightarrow$ $\lim_{n\to\infty} a_n = 0$

(taking $m = n$ in previous thm)

but the converse is <u>not true</u>.

e.g. the harmonic series

$\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge

Thm A series of non-negative terms converges iff the partial sums are bounded

(partial sums are monotone sequence)

Thm (a) If $|a_n| \le c_n$ for $n \ge N_0$, some fixed integer, and $\sum' c_n$ converges then $\sum' a_n$ converges.

(b) If $a_n \ge d_n \ge 0$ for $n \ge N_0$ and $\sum' d_n$ diverges then $\sum' a_n$ diverges.

proof: use Cauchy criterion, $\forall\ \varepsilon > 0\ \exists\ N \ge N_0$ s.t. $\forall\ n \ge m \ge N$

$$\varepsilon \ge \left|\sum_{k=m}^{n}{}' c_n\right| = \sum_{k=m}^{n}{}' c_n \ge \sum_{k=m}^{n}{}' |a_n| \ge \left|\sum_{k=m}^{n}{}' a_n\right|$$

$\square$