## Power Set

For any set $X$ we define the power set

$$2^X := \{E : E \subset X\}$$ the set of all subsets of $X$.

e.g. if $X = \{1, \dots, n\}$ finite

What is the cardinality of $2^X$?

~~for each E~~ $E \subset X$ can be encoded by a binary string of length $n$

$$a_1 a_2 \cdots a_n, \quad a_j = \begin{cases} 1 & \text{if } j \in E \\ 0 & \text{if } j \notin E \end{cases}$$

There are $2^n$ such strings.

**Thm (Cantor?)** For any set $X$ the cardinality of $2^X$ > cardinality of $X$.

proof: there is an easy injection $f: X \to 2^X$

$f(x) = \{x\}$ so $\mathrm{card}(X) \leq \mathrm{card}(2^X)$

suppose that there is a bijection $f: X \to 2^X$

define $E = \{x \in X : x \notin f(x)\}$

then $E = f(y)$ for some $y \in X$

suppose $y \in f(y)$ then $y \notin f(y)$ $\rightarrow\leftarrow$

suppose $y \notin f(y)$ then $y \in f(y)$ $\rightarrow\leftarrow$

so $y$ cannot be in $f(y)$ or in $2^X \setminus f(y)$

$\rightarrow\leftarrow$ $\square$

Cor: $2^{\mathbb{N}}$ is uncountable.

# Connected Sets

let $(X,d)$ metric space, $E \subset X$

We say that a pair $U, V$ is

an open cut / partition of $E$

if $U, V$ disjoint, ~~and~~ $E \subset U \cup V$ and

$E \cap U \neq \phi$ and $E \cap V \neq \phi$

We say $E$ is <u>connected</u> if there is no

open cut of $E$.

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<u>Thm</u> $X$ is connected if and only if

every set $E \subset X$ which is both closed and open is $= X$ or $\phi$.

<u>proof</u>: Suppose $X$ connected.

Suppose $E \subsetneq X$ nonempty is closed and open

then $E^c$ is open and

$E, E^c$ is an open cut of $X$

$\Rightarrow$ $X$ is not connected $\rightarrow\leftarrow$.

Suppose every clopen set of $X$ is $= X$ or $\emptyset$

suppose $U, V$ is an open cut of $X$.

so $X \subset (X \cap U) \cup (X \cap V) = \{U \cup V \subset X$

so since $U, V$ disjoint and

$X = U \cup V \Rightarrow V = U^c$

$\Rightarrow$ $V$ clopen $\Rightarrow$ $U$ clopen

so either $U$ or $V$ is empty

contradicting that $X \cap U, X \cap V \neq \emptyset$.

$\rightarrow\leftarrow$ $\square$

<u>Thm</u> $E \subset \mathbb{R}$ is connected iff $E$ is an interval

<u>proof</u>: Suppose $E \subset \mathbb{R}$ is connected we show

that if $x, y \in E$ then $z \in E$ for all $x < z < y$.

then ~~$E \subset [\inf E, \sup E]$~~,

$(\inf E, \sup E) \subset E \subset [\inf E, \sup E]$.

Suppose $x < z < y$ and $z \notin E$

then $(-\infty, z)$, $(z, +\infty)$ is

an open cut of $E$. $\to\Leftarrow$.

$(\Leftarrow)$

Suppose $E$ is an interval

and $U, V$ is an open cut of $E$

let $x \in U \cap E$
$y \in V \cap E$

$x < y$ (or vice versa)

Since $E$ interval $[x, y] \subset E$

Call $z = \sup U \cap [x, y] < y$
since $y \in V$

$z \notin U$ since otherwise by $U$ open
wouldn't be the sup

and $z \notin V$ since by $V$ open
couldn't be a limit pt of $U$

so $z \notin E$ $\to\Leftarrow$.