ected

A path from x to y in a metric space X is a cts function $\gamma : [0,1] \to X$ s.t.

$$\gamma(0) = x, \quad \gamma(1) = y.$$

A space X is called path connected if for every $x, y \in X$ $\exists$ a path γ in X from x to y.

Thm ~~If~~ $E \subseteq \mathbb{R}^k$ is open then E is path connected iff E is connected.

proof: Suppose U ~~is path connected~~ not connected

i.e. $\exists$ U, V open ~~and~~ disjoint

w/ $U \cap E \neq \phi$, $V \cap E \neq \phi$.

Let $x \in U \cap E$ and $y \in V \cap E$

and γ a path from x to y

since the image of connected set under

cts fcn is connected $\gamma([0,1])$

is connected, but U, V

separate $\gamma([0,1])$ as well since

$x \in U \cap \gamma([0,1])$ and $y \in V \cap \gamma([0,1])$

$\Rightarrow\Leftarrow$ so E <u>not</u> path connected.

Now suppose E is connected and

$\exists$ $x, y \in E$ with ~~not~~ no cts

path in E between them.

Call $U = \{z \in E : \exists$ path from x to z in $E\}$

$V = \{z \in E : \exists$ path from y to z in $E\}$

$W = \{z \in E : z$ does not have a path to x or y in $E\}$

Claim U, V, W are open, disjoint

thus ~~$U \cup V$~~ $U \cup W$ and V would

separate E (which is a contradiction)

(1) U open

suppose $z \in E$ so $\exists$ path γ from

x to z.

let $\delta > 0$ small enough that $B(z, \delta) \subset E$

(since $E \subset \mathbb{R}^2$ open)

for $z' \in B(z, \delta)$ define

$$\gamma'(t) = \begin{cases} \gamma\left(\frac{t}{1-\delta}\right) & \text{for } 0 \le t \le 1-\delta \\ z + (t-(1-\delta))\frac{(z'-z)}{|z'-z|} & \text{for } 1-\delta \le t \le 1 \end{cases}$$

which is now a cts path from x to z'

$\Rightarrow z' \in U \Rightarrow B(z,\delta) \subset U$

so U is open. $\blacksquare$

Same argument $\Rightarrow$ V is open

~~Since~~ ~~$E = U \cup V \cup W$~~

~~all disjoint and E open~~

for W, since E is open ~~by~~

$\forall z \in W$ $\exists$ $\delta > 0$ s.t. $B(z,\delta) \subset E$

suppose $B(z,\delta) \cap U$ or $B(z,\delta) \cap V$

nonempty then we could make

a path in E from x or y resp

to z as before which

contradicts $z \in W$ $\Rightarrow$ $B(z,\delta) \subset W$.

~~Similar~~

Similar argument for U, V, W disjoint.

If $z \in U \cap V$ then $\exists$ γ_1, γ_2 s.t.

γ_1 path ~~x~~ x to z, γ_2 path y to z

$$\gamma(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ \gamma_2(1-2(t-1/2)) & \frac{1}{2} \le t \le 1 \end{cases}$$

is a path from x to y which we assumed did not exist.

So $U \cup W$, V separate E

which is a contradiction.

$\square$

<u>Example</u> Metric space which is connected but not path connected.

Call $X =$ the closure in $\mathbb{R}^2$ of

$$\{(x, \sin\tfrac{1}{x}) : x \in \mathbb{R} \setminus \{0\}\}$$

with the Euclidean metric.