Metric Spaces

A set X (w/ elements called <u>points</u>)

~~is called~~

w/ a function $d: X \times X \to \mathbb{R}$

$d(p,q)$ called the distance from p to q

is called a metric space if d satisfies

(a) $d(p,q) > 0$ if $p \neq q$, $d(p,p) = 0$
(b) $d(p,q) = d(q,p)$
(c) $d(p,q) \leq d(p,r) + d(r,q)$ for all $r \in X$

for every $p, q \in X$.

d is called a <u>metric</u> or <u>distance function</u>

<u>Examples</u>

$\mathbb{R}, \mathbb{R}^2, \mathbb{R}^n$ with

metric $d(x,y) = |x-y| = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$.

(we already know this satisfies triangle inequality)

[Other examples e.g. function space

$C(K) = \{ f : K \to \mathbb{R} : f \text{ continuous} \}$

$d(f,g) = \sup_{x \in K} |f(x) - g(x)|$.]

discrete metric (see HW)

Define let (X, d) a metric space

(a) a neighborhood or ball centered at p

is the set $N_r(p)$ or $B(p,r)$

defined by $\{ q \in X : d(p,q) < r \}$

$r > 0$ is called the radius

(b) a limit point of a set $E \subset X$ is

a point $p \in X$ s.t. for every

$r > 0$ $\left(B(p,r) \setminus \{p\} \right) \cap E \neq \emptyset$.

examples $(\mathbb{R}, |\cdot\cdot\cdot|)$

open interval (a, b)

$$\{\tfrac{1}{n} : n \in \mathbb{N}_{+}\}$$

(c) $p \in E$ and p not a limit point of E

then p called an isolated point

(d) E is called closed if every limit point of E is contained in E.

e.g. $(0,1]$ $\to$ not closed
$[0,1]$ $\to$ closed

(under Euclidean metric)

(e) p is an interior pt of E if

$\exists$ $r > 0$ s.t. $B(p,r) \subset E$.

(f) E called open if every $p \in E$ is an interior point

(g) the complement of E (called E^c)

$= \{p \in X : p \notin E\}$

(h) E is *perfect* if E is closed and every point of E is a limit point of E.

e.g. $[0,1] \to$ perfect
$[0,1] \cup \{2\} \to$ not perfect

(i) E is *bounded* if $\exists$ $M>0$ ~~s.t.~~ and $q \in E$ s.t.

$$E \subseteq B(q, M).$$

(j) E is *dense* in X if every point in X is ~~the~~ a limit point of E.

examples ~~X dense in~~
$\mathbb{Q}$ dense in $\mathbb{R}$.

(k) Call $E' = \{\text{limit points of } E\}$

$\bar{E} = E \cup E'$ *closure of E*

$E^\circ = \{\text{interior points of } E\}$ *interior of E*

Thm $B(p,r)$ is open

proof: let $q \in B(p,r)$ then

$$d(q,p) < r$$

let $h>0$ s.t. $0 < h < r - d(q,p)$

then claim $B(q,h) \subset B(p,r)$

let $r \in B(q,h)$ then

$$\begin{aligned} d(r,p) &\le d(r,q) + d(q,p) \\ &< h + d(q,p) \\ &< r - d(q,p) + d(q,p) = r. \end{aligned}$$

Thm If p is a limit pt of E then $E \cap B(p,r)$ is infinite for every $r>0$.

proof: suppose $\exists\ r>0$ s.t. $B(p,r) \cap E$ is finite

enumerate its elements $q_1, \dots, q_n, p$

~~let~~ put $r = \min\limits_{j=1,\dots,n} d(q_j,p) > 0$

minimum of a finite set of positive numbers is positive.

then $\left(B(p,r) \setminus \{p\}\right) \cap E = \phi$

$\Rightarrow$ p not a limit pt of E

$\rightarrow\leftarrow$. $\square$

<u>Thm</u> (De-Morgans Law)

let $\{E_\alpha\}_{\alpha \in A}$ be a collection of sets then

$$\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c$$

<u>proof</u>: Let $A = \left(\bigcup_\alpha E_\alpha\right)^c$, $B = \bigcap_\alpha E_\alpha^c$

if $x \in A$ ~~then~~ $x \notin E_\alpha$ for any α

hence $x \in E_\alpha^c$ for every $\alpha \Rightarrow x \in B$

if $x \in B$ then $x \in E_\alpha^c$ for every α

then $x \notin E_\alpha$ for any α then $x \in A$.

Thm A set E is open iff E^c is closed

proof: suppose E^c is closed

let $x \in E$, then $x \notin E^c$

so x not a limit point of E^c

so $\exists\, r > 0$ s.t. $B(x,r) \cap E^c = \phi$

i.e. $B(x,r) \subset E$.

Suppose E is open

let x a limit pt of E^c.

then $B(x,r) \cap E^c \neq \phi$ for every $r > 0$

so x not an interior point of E

$\Rightarrow$ $x \notin E$ (since E open)

so $x \in E^c$.

Thm (a) for any collection $\{G_\alpha\}$ of open sets $\bigcup_\alpha G_\alpha$ open

(b) for any collection $\{F_\alpha\}$ of closed sets $\bigcap_\alpha F_\alpha$ closed

(c) for any *finite* collection $\{G_n\}_{n=1}^m$ open

$\cancel{\bigcup_{n=1}^m G_n \text{ open}}$ $\bigcap_{n=1}^m G_n$ is open

(d) for any *finite* collection $\{F_n\}_{n=1}^m$ closed

$\bigcup_{n=1}^m F_n$ closed