Linear Space (Vector Space)

over field $\mathbb{R}$ (or $\mathbb{C}$ or more general)

is a space V w/ addition and scalar mult

for $u, v, w \in V$ and $a, b \in \mathbb{R}$ (or $\mathbb{C}$ or F)

(i) $u + (v + w) = (u + v) + w$ (assoc.)

(ii) $u + v = v + u$ (commut.)

(iii) $\exists\ 0 \in V$ s.t. $0 + u = u \quad \forall\ u \in V$

(iv) $\forall\ u \in V \quad \exists\ -u \in V$ s.t.

$$u + (-u) = 0$$

(v) $a(bv) = (ab)v$

(vi) $1v = v$

(vii) $a(u+v) = au + bv$

(viii) $(a+b)v = av + bv$.

Examples

- $\mathbb{R}^k$ for $k \in \mathbb{N}_+$
- $\mathbb{R}^{\mathbb{N}_+} = \{ (a_1, a_2, \ldots) : a_i \in \mathbb{R} \}$
- $C(K) = \{ f : K \to \mathbb{R} : f \text{ is cts} \}$ w/ K cpct metric space.

~~Banach Space~~

Normed Space

A normed linear space is a linear space V

w/ a norm $\|\cdot\| : V \to [0,+\infty)$

satisfying

(i) $\|x\| > 0$ iff $x \neq 0$

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for $\alpha \in \mathbb{R}$

(iii) $\|x+y\| \leq \|x\| + \|y\|$ Δ-ineq

Rem: Every normed linear space V is a metric space w/ metric

$$d(x,y) = \|x-y\|$$

(i) $\|x-y\| = 0$ iff $x-y=0$

(ii) $\|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\|$.

Examples:

- $\mathbb{R}^k$ w/ $\|x\|_2 = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2}$
- $\mathbb{R}^k$ w/ $\|x\|_p = \left(\sum_{i=1}^{k} x_i^p\right)^{1/p}$

  for any $1 \leq p < \infty$

  $\|x\|_\infty = \max\{|x_1|, \dots, |x_k|\}$

- $\mathbb{R}^{\mathbb{N}_+}$ w/ $\|x\|_p = \left(\sum_{i=1}^{\infty} x_i^p\right)^{1/p}$ $1 \leq p < \infty$

  $\|x\|_\infty = \sup_{i \in \mathbb{N}_+} |x_i|$

  Called $\ell^p(\mathbb{N}_+)$ space.

- $C(K)$ w/ $\|f\| = \sup_{x \in K} |f(x)|$

  which is finite because cts fcns on compact sets are bounded

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~~proof~~ Thm (Hölder's Ineq) If $p \geq 1$, $q \geq 1$ s.t.

$\frac{1}{p} + \frac{1}{q} = 1$ then for any sequences

$x_k, y_k \in \mathbb{R}$

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \left(\sum |x_k|^p\right)^{1/p} \left(\sum |y_k|^q\right)^{1/q}$$

proof: if $\|x\|_p$ or $\|y\|_q = +\infty$ it is ~~obs~~ obvious

if $\|x\|_p$ or $\|y\|_q = 0$ also obvious.

now ~~#~~ by looking at

$$x'_k = \frac{x_k}{\|x\|_p} \qquad y'_k = \frac{y_k}{\|y\|_q}$$

can assume that $\|x\|_p = 1$, $\|y\|_q = 1$

then use Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} = 1$$

(proof: using that log is concave

$$\log\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \geq \frac{1}{p}\log a^p + \frac{1}{q}\log b^q$$

(since $\frac{1}{p} + \frac{1}{q} = 1$)

$$= \log(ab)$$)

$$\sum_{k=1}^{N} |x_k y_k| \leq \sum_{k=1}^{N} \frac{|x_k|^p}{p} + \frac{|y_k|^q}{q} = \frac{1}{p}\sum_{k=1}^{N} |x_k|^p + \frac{1}{q}\sum_{k=1}^{N} |y_k|^q$$

$$\leq \frac{1}{p} + \frac{1}{q} = 1$$

so $$\sum_{k=1}^{\infty} |x_k y_k| \leq 1$$

Thm (Minkowski's Ineq) for $x, y \in \ell^p(\mathbb{N}_+)$ or $\ell^p(\{1, \dots, k\})$

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

proof:

$$\sum_{k=1}^{N} |x_k + y_k|^p = \sum_{k=1}^{N} |x_k + y_k| |x_k + y_k|^{p-1}$$

$$\leq \sum_{k=1}^{N} \left(|x_k| + |y_k| \right) |x_k + y_k|^{p-1}$$

$$= \sum_{k=1}^{N} |x_k| |x_k + y_k|^{p-1} + \sum_{k=1}^{N} |x_k| |x_k + y_k|^{p-1}$$

(Hölder Ineq w/ p and $\frac{1}{q} = 1 - \frac{1}{p}$ i.e. $q = \frac{p}{p-1}$)

$$\leq \left(\sum_{k=1}^{N} |x_k|^p \right)^{1/p} \left(\sum_{k=1}^{N} |x_k + y_k|^{\frac{p}{p-1}(p-1)} \right)^{\frac{p-1}{p}}$$

$$+ \left(\sum_{k=1}^{N} |y_k|^p \right)^{1/p} \left(\sum_{k=1}^{N} |x_k + y_k|^p \right)^{1-\frac{1}{p}}$$

So dividing through by $\left(\sum_{k=1}^{N} |x_k + y_k|^p \right)^{1-\frac{1}{p}}$

$$\left(\sum_{k=1}^{N} |x_k + y_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^{N} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^{N} |y_k|^p \right)^{1/p}$$

then let $N \to \infty$ on left and right sides.

So ℓ^p spaces are indeed normed linear spaces. In particular they are metric spaces as well.

Lets show that

Thm $\overline{B(0,1)} = \{ x \in \ell^p(\mathbb{N}) : \|x\|_p \le 1 \}$ in $\ell^p(\mathbb{N})$ is <u>not</u> compact.

<u>proof:</u> I'll show that there is an infinite sequence w/ no convergent subsequence.

Let $\cancel{x_n}$ $x_k^n = \begin{cases} 0 & n \neq k \\ 1 & n = k \end{cases}$

i.e. $x^1 = (1, 0, \dots)$
$x^2 = (0, 1, 0, \dots)$
$x^3 = (0, 0, 1, 0, \dots)$

All x^n have $\|x^n\|_p = 1$

but also $\|x^n - x^m\| = 1$ for all $n \neq m$

$\Rightarrow$ x^n not Cauchy

$\Rightarrow$ x^n does not converge.

$\square$

Closed and bdd sets are not ~~always~~ compact ~~ex~~ in general.

A Banach Space is a complete normed linear space.

Thm $\ell^p(\mathbb{N})$ is ~~complete for~~ a Banach space for every $1 \leq p < \infty$.

proof: let x^n Cauchy in $\ell^p(\mathbb{N})$

since $|x^n(i) - x^m(i)| \leq \|x^n - x^m\|_p$

Can show $x^n(i)$ Cauchy in n for all $i \in \mathbb{N}$.

Call $\quad x(i) = \lim_{n\to\infty} x^n(i)$ which exists since $\mathbb{R}$ is complete

Claim that $\quad x^n \to x \quad$ in ℓ^p

~~Let $\varepsilon > 0$. $\exists$ K s.t.~~

$$\cancel{\left(\sum_{k \geq K} |x_k|^p\right)^{1/p} \leq \varepsilon}$$

~~First let's show that x is actually in ℓ^p~~

$$\cancel{\text{i.e. } \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} < +\infty}.$$

$\forall \; \varepsilon > 0 \quad \exists \; N \quad$ s.t. $\quad \forall \; n, m \geq N$

$$\|x^n - x^m\|_{\ell^p}^p = \sum_{j=1}^{\infty} |x_j^n - x_j^m|^p < \varepsilon$$

Since partial sums are monotone $\forall \; M \in \mathbb{N}$

$$\sum_{j=1}^{M} |x_j^n - x_j^m|^p < \varepsilon \qquad \text{for } n, m \geq N$$

then send $m \to \infty$ since LHS is finite sum of cts fcns $\left(|x_j^n - \cdot|^p \text{ is cts}\right)$

$$\sum_{j=1}^{M} |x_j^n - x_j|^p < \varepsilon \qquad \text{for every } M \in \mathbb{N},\ n \geq N$$

then send $M \to \infty$, ε is an UB for partial sums so their limit

(which is their sup since terms are positive $\Rightarrow$ sequence of partial sums is monotone incr.)

$$\sum_{j=1}^{\infty} |x_j^n - x_j|^p < \varepsilon \qquad \text{for all } n \geq N$$

$$\| x^n - x \|_p < \varepsilon^{1/p} \qquad \text{for all } n \geq N.$$

this means that $x \in \ell^p(\mathbb{N})$ since taking $\varepsilon = 1$ $\exists N$ s.t.

$$\|x\|_p \leq \|x - x^N\|_p + \|x^N\|_p \leq 1 + \|x^N\|_p < +\infty$$

and $x^n \to x$ in ℓ^p metric

thus every Cauchy sequence in $\ell^p(\mathbb{N})$ converges

$\Rightarrow$ $\ell^p(\mathbb{N})$ is complete. $\square$

Thm $C(K)$ w/ sup norm is complete

(i.e. it is a Banach space)

proof: Let f_n be a Cauchy sequence in $C(K)$

$\forall$ $\varepsilon > 0$, $\exists$ N s.t. $n, m \geq N$

$$\Rightarrow \quad \|f_n - f_m\|_{C(K)} = \sup_{t \in K} |f_n(t) - f_m(t)| < \varepsilon$$

so in particular since for any fixed $t \in K$

$$|f_n(t) - f_m(t)| \leq \|f_n - f_m\|_{C(K)} < \varepsilon$$

for $n, m \geq N$ as above

$\Rightarrow$ $f_n(t)$ Cauchy in $\mathbb{R}$

so it converges, call the limit $f(t)$.

Let $\varepsilon > 0$, $\exists$ N s.t. for all $n, m \geq N$

$$|f_n(t) - f_m(t)| < \varepsilon \quad \text{for all } t \in K$$

take the limit as $m \to \infty$

$$|f_n(t) - f(t)| \leq \varepsilon \quad \text{for all } t \in K, \; n \geq N.$$

So in other words

$$\|f_n - f\|_{C(K)} = \sup_{t\in K} |f_n(t) - f(t)| < \varepsilon \quad \text{for } n \geq N.$$

$\Rightarrow$ $f_n \to f$ in $\|\cdot\|_{C(K)}$ norm.

As we will show later this actually means that f is a continuous function as well. $\square$

Thm If $\|f_n - f\|_{C(K)} \to 0$ as $n \to \infty$, $f_n \in C(K)$, $f: K \to \mathbb{R}$ then f is continuous on K.

proof: let $x \in K$, $\varepsilon > 0$. $\exists$ N s.t.

$$n \geq N \Rightarrow \|f_n - f\|_{C(K)} < \frac{\varepsilon}{3}$$

f_N cts $\Rightarrow$ $\exists$ $\delta > 0$ s.t. $d(x,y) < \delta$

$$\Rightarrow |f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$$

then $|y-x| < \delta \Rightarrow$

$$|f_m(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon \quad \square$$

<u>Space which is Not Complete</u>

We could put a different norm on ~~$C(K)$~~ $C([0,1])$ called the ~~$L^1(K)$~~ $L^1([0,1])$ norm

$$\|f\|_{L^1([0,1])} := \int_0^1 |f(x)|\,dx$$

This normed space ~~$(C(K), \|\cdot\|_L$~~

$$\left(C([0,1]), \|\cdot\|_{L^1([0,1])}\right)$$

is <u>not</u> a Banach space because it is not <u>complete</u>.

'' Lets do on $[0,2]$ actually

proof: Take $f_n(t) = \begin{cases} t^n & 0 \le t \le 1 \\ 1 & 1 \le t \le 2 \end{cases}$

idea:
as $n \to \infty$ $t^n \to 0$ for every $t \in [0,1)$

so if it were to converge

$$f_n \to h, \quad h(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \end{cases}$$

but that function is not cts.

Lets show that f_n is in fact Cauchy in the L^1 metric.

$$\|f_n - f_m\|_{L^1([0,2])} = \int_0^1 |t^n - t^m|$$

take $n < m$, WLOG

$$= \int_0^1 (t^n - t^m)\,dt$$

$$= \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n+1} \to 0 \text{ as } n \to \infty.$$

Now lets show that f_n cannot possibly converge to any cts fn in L^1 metric....

skip being more careful about integration since we don't know yet anyway.

Another Examples

Let $\quad l_0(\mathbb{N}) = \{ (a_1, a_2, \dots) : \text{only finitely many } a_j \neq 0 \}$

Can make $l_0(\mathbb{N})$ into a normed linear space with any of the l^p norms

$$(l_0(\mathbb{N}), \|\cdot\|_p) \qquad 1 \le p < \infty$$

This space is not complete

e.g. given $x \in l^p(\mathbb{N})$ take

$$x^n = (x_1, x_2, \dots, x_n, 0, \dots) \in l_0(\mathbb{N})$$

for $n < m$,

$$\|x^n - x^m\|_p = \left(\sum_{k=n}^{m} |x_k|^p \right)^{1/p}$$

Let N suff large so that $\forall\, n \geq m \geq N$

$\sum_{k=n}^{m} |x_k|^p < \varepsilon^p$ (allowed since sum converges)

then for $n > m \geq N$ $\quad \|x^n - x^m\|_p < \varepsilon$ $\square$

On the other hand $l_0(\mathbb{N})$ is dense in every $\ell^p(\mathbb{N})$

Thm $l_0(\mathbb{N})$ is a dense ~~subset~~ subspace of $\ell^p(\mathbb{N})$.

proof: let $x \in \ell^p(\mathbb{N})$ and $\varepsilon > 0$

$$\exists\ N \quad \text{s.t.} \quad \left(\sum_{N+1}^{\infty} |x_k|^p \right)^{1/p} < \varepsilon$$

take $\quad y = (x_1, \dots, x_N, 0, \dots) \in l_0(\mathbb{N})$

$$\|y - x\|_{\ell^p} \le \left(\sum_{N+1}^{\infty} |x_k|^p \right)^{1/p} < \varepsilon \quad \square$$

~~Note that $\ell^p(\mathbb{N})$ is not countable~~

but $l_0(\mathbb{N})$ is countable

so $\ell^p(\mathbb{N})$ has a countable dense subspace i.e. it is a separable metric space.

Non-separable Metric Space

$$\ell^\infty(\mathbb{N}_+) = \left\{ x \in \mathbb{R}^{\mathbb{N}_+} : \sup_i |x_i| < +\infty \right\}$$

with norm $\|x\|_\infty = \sup_{i \in \mathbb{N}_+} |x_i|$

Δ - ineq -

$$\begin{aligned}
\|x+y\|_\infty &= \sup \left\{ |x_i + y_i| : i \in \mathbb{N}_+ \right\} \\
&\le \sup \left\{ |x_i| + |y_i| : i \in \mathbb{N}_+ \right\} \\
&\le \sup_i |x_i| + \sup_i |y_i| \\
&= \|x\|_\infty + \|y\|_\infty .
\end{aligned}$$

$\ell^\infty(\mathbb{N}_+)$ is a Banach space but it is <u>not</u> separable

Thm $\ell^\infty(\mathbb{N}_+)$ has no countable dense subset.

proof: This will be like the Cantor diagonal argument

suppose ~~$\{X_i\}_{i=1}^{\infty}$ is a den~~ $\{x^\alpha\}_{\alpha=1}^{\infty}$ is a countable dense set of $\ell^\infty(\mathbb{N}_+)$.

We will construct a point y s.t.

$$\|x^\alpha - y\|_\infty = 1 \quad \text{for all } \alpha = 1, 2, \ldots$$

so then $B(y,1) \cap \{x^\alpha : \alpha \in \mathbb{N}_+\} = \emptyset$

which contradicts that ~~x_α~~ x^α are dense.

each ~~let~~

[illegible]

for $i \in \mathbb{N}_+$ define $\boxed{y_i = x_i^i + 1}$ so that

$$1 = |y_i - x_i^i| \leq \sup_j |y_j - x_j^i| = \|x^i - y\|_\infty$$

so $\|x^\alpha - y\|_\infty = 1 \quad \forall \alpha \in \mathbb{N}_+$.

$\square$