MATH 203: Homework 7

Due Wednesday Nov 30

Problems are from Rudin 3rd edition.

Problem 1. Chapter 2 (p. 44): 17, 18, 21

Problem 2. Chapter 3 (p. 79): 16, 19

Problem 3. Chapter 4 (p. 98): 6, 9, 11, 25

Problem 4. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in a metric space X with the following property: there is $x \in X$ such that every subsequence of the x_n has a subsequence converging to x. Show that x_n converges to x.

Problem 5. Given a function $f : \mathbb{R}^k \to \mathbb{R}$ define the *support of f*

$$\mathrm{supp}(f) = \overline{\{x \in \mathbb{R}^k : \ |f(x)| > 0\}}.$$

A function f is said to have *compact support* if $\mathrm{supp}(f)$ is a compact set of $\mathbb{R}^k$. We define the vector space of continuous functions with compact support on $\mathbb{R}^k$ (why is it a vector space? check for yourself you don't need to write it):

$$C_c(\mathbb{R}^k) = \{f : \mathbb{R}^k \to \mathbb{R} : \ f \text{ is continuous and has compact support}\}.$$

Show that $C_c(\mathbb{R}^k)$ with the norm $\|f\| := \sup_{x \in \mathbb{R}^k} |f(x)|$ is a normed space (i.e. check that $\|f\| < +\infty$ for all $f \in C_c(\mathbb{R}^k)$). Show by an example that this normed space is *not* complete.

The following final part is just for your own interest you don't have to do it: show that the completion of $C_c(\mathbb{R}^k)$ under the supremum norm $\| \cdot \|$ is the space $C_0(\mathbb{R}^k)$ of continuous function which vanish at ∞,

$$C_0(\mathbb{R}^k) := \{f : \mathbb{R}^k \to \mathbb{R} : \ f \text{ is continuous and } \lim_{|x| \to \infty} f(x) = 0\}.$$

Problem 6. Two metrics d_1 and d_2 on a space X are said to be *equivalent* if for any sequence $\{x^j\}_{j=1}^{\infty}$ in X,

$$x^j \text{ converges to } x \text{ in } d_1 \text{ metric if and only if } x^j \text{ converges to } x \text{ in } d_2 \text{ metric.}$$

(i) Show that all the ℓ^p distances for $1 \le p \le \infty$ on $\mathbb{R}^k$ are equivalent. Recall that

$$d_{\ell^p}(x, y) = \left(\sum_{n=1}^{k} |x_n - y_n|^p \right)^{1/p} \quad \text{and} \quad d_{\ell^\infty}(x, y) = \sup_{n=1,\dots,k} |x_n - y_n|.$$

Hint: Show that convergence in any d_{ℓ^p} is equivalent to convergence of all the entries in absolute value.

(ii) Now consider the sequence space,

$$\ell^1(\mathbb{N}_+) = \{x = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}_+} : \ \|x\|_1 = \sum_{n=1}^{\infty} |x_n| < +\infty\}.$$

Show that convergence with respect to ℓ^1-metric $d_{\ell^1}(x, y) = \|x - y\|_1$ implies convergence with respect to the ℓ^∞ metric

$$d_{\ell^\infty}(x, y) := \sup_{n \in \mathbb{N}_+} |x_n - y_n|.$$

Give an example of a sequence in the space $\ell^1(\mathbb{N}_+)$ which converges in ℓ^∞ metric but not in ℓ^1 metric.

Problem 7. [Tao, *Analysis II*, 12.5.10] A metric space (X, d) is called *totally bounded* if for every $\varepsilon > 0$ there exists a positive integer n and points $x_1, \dots, x_n \in X$ so that $B(x_1, \varepsilon), \dots, B(x_n, \varepsilon)$ cover X.

(i) Show that a totally bounded space is bounded.

(ii) Show that if (X, d) is sequentially compact (every sequence has a convergent subsequence) then it is complete and totally bounded. Hint: if X is not totally bounded then there is some $\varepsilon > 0$ so that there is no finite covering of X by balls of radius ε. Show that in this situation, given any finite collection of $\varepsilon/2$ radius balls there is another $\varepsilon/2$ radius ball which is disjoint from all of them. Use this inductively to construct an infinite sequence $B(x_j, \varepsilon/2)$ of disjoint balls. From here you can contradict sequential compactness.