Finite, Countable & Uncountable Sets

Def: Let $A, B$ be sets and suppose that for each element $x \in A$ there is associated an element of $B$, $f(x)$. Then $f$ is said to be a function (or mapping of $A$ into $B$)

from $A$ to $B$. The set $A$ is called the domain of $A$, ~~set B is call~~ the elements

$f(x) \in B$ are called the values of $f$, and

the set of all values of $f$

$$\{y \in B : y = f(x) \text{ for some } x \in A\}$$

is called the range of $f$.

We write $f : A \to B$.

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Let $f : A \to B$. ~~we call~~ Let $E \subset A$

we call $f(E) = \{ y \in B : y = f(x) \text{ for some } x \in E \}$

is called the image of $E$ under $f$.

$f(A)$ is the range of $f$.

$f(A) \subset B$ by definition. ie

$f(A) = B$ we say $f$ is maps $A$ onto $B$

(sometimes called surjective)

For ~~$E \subset B$~~ $E \subset B$

$f^{-1}(E) = \{x \in A : f(x) \in E\}$

Called the inverse image of $E$ under $f$.

for $y \in B$, $f^{-1}(y)$ is a subset of $A$

so that $\forall$ $x \in f^{-1}(y)$, $f(x) = y$.

If for all $y \in B$ $f^{-1}(y)$ consists of at most one element (could be zero)

the we say $f$ is a 1-1 (one-to-one) mapping of $A$ into $B$.

(also called injective).

Note: (equivalently)

$f$ is injective iff $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1, x_2 \in A$.

Def If there is a bijective (injective & surjective) mapping of $A$ onto $B$ we say that $A$ and $B$ "can be put in 1-1 correspondence"

or $A$ and $B$ have the same cardinality

write $A \sim B$.

reflexive: $A \sim A$
symmetric: $A \sim B \iff B \sim A$
transitive: $A \sim B$ and $B \sim C \Rightarrow A \sim C$.

easy exercise: prove those properties.

Def: For any set $A$ we say
(i) $A$ is finite if $A \sim \{1, \dots, n\}$ for some $n \in \mathbb{N}$
(ii) $A$ is infinite if $A$ not finite
(iii) $A$ is countable if $A \sim \mathbb{N}$
(iv) $A$ is uncountable if not countable
(e) $A$ is (at most) countable if $A$ finite or countably infinite.

for finite sets $A \sim B$ iff
A and B have the same number of elements

<u>Example</u> $\mathbb{Z}$, the integers, is a countable set.

construct a bijection $f: \mathbb{Z} \to \mathbb{N}$

map negative #'s to the odd ~~set~~ naturals

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n-1 & \text{if } n < 0 \end{cases}$$

$\mathbb{Z}$: 0, -1, 1, -2, 2, ...
$\mathbb{N}$: 0, 1, 2, 3, 4, ...

$f$ is 1-1 and onto.

This example is a bit counter-intuitive since finite sets cannot be equivalent to a proper subset of themselves (different cardinality), but for infinite sets it is possible.

<u>Def</u>: A sequence is a function on the positive integers (range unspecified). We usually denote

$f(n) = x_n$ and think of the sequence $\{x_n\}_{n=1}^{\infty}$ or as $x_1, x_2, x_3, \ldots$ . The $x_n$ are called the terms in the sequence. ~~if f~~

If $A$ is a set and $x_n \in A$ for all $n$ then $x_n$ is a <u>sequence of elements of $A$</u>

or a <u>sequence in $A$</u>

~~Since~~ Note that we can thus think of a countable set $A$ as the range of a sequence of distinct elements of $A$.

**Thm:** Every infinite subset $B$ of a countable set $A$ is countable.

<u>Proof</u>: there is a bijection $f: \mathbb{N} \to A$

let $n_1$ be the smallest $n > 0$ s.t. $f(n) \in B$

$\vdots$

$n_k$ be the smallest $n > n_{k-1}$ s.t. $f(n) \in B$

~~results~~ since $B$ is infinite this process does not terminate, we get a sequence

$$g(k) = f(n_k) \quad , \quad g: \{1, 2, \ldots\} \to B .$$

it is 1-1 since $f$ was 1-1.

it is onto since $f$ was onto $A$

(for all $x \in B$ $\exists$ $n_x$ s.t. $f(n_x) = x$
we know that $n_x = n_k$ for some $k \leq n_x$.) $\square$

~~Def~~ Let $A$ be a set
(called <u>index set</u> in this context)

and $\Omega$ a set s.t. for each $\alpha \in A$

we associate an $E_\alpha \subset \Omega$.

$\{E_\alpha : \alpha \in A\}$ is a <u>collection</u> or <u>family</u> of subsets of $\Omega$.

<u>define</u> the <u>union</u> of the family $S$

$x \in S$ iff $x \in E_\alpha$ for at least one $\alpha \in A$

or write

$$S = \bigcup_{\alpha \in A} E_\alpha .$$

If (a typical situation) $A = \{1, \ldots, n\}$ we write

$$S = \bigcup_{m=1}^{n} E_m .$$

if $A = \mathbb{N}$ then write

$$S = \bigcup_{n=1}^{\infty} E_n .$$

The <u>intersection</u> of the $E_\alpha$

is the $x$ s.t. $x$ in <u>every</u> $E_\alpha$ $\alpha \in A$

$$\bigcap_{\alpha \in A} E_\alpha = \{x \in \Omega : x \in E_\alpha \text{ for all } \alpha \in A\}$$

if $A, B$ sets $A \cap B = \phi$ say $A$ and $B$

are <u>disjoint</u>.

Rules ($\cup$ like $+$, $\cap$ like $\cdot$)
Algebra
$A \cup B = B \cup A$ $\qquad$ $A \cap B = B \cap A$
$A \cup (B \cup C) = (A \cup B) \cup C$ $\qquad$ $(A \cap B) \cap C = A \cap (B \cap C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ <u>distributive</u>

let $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in B \cup C$

i.e. $x \in A$ and ($x \in B$ <u>or</u> $x \in C$) (or both

so $x \in A \cap B$ <u>or</u> $x \in A \cap C$.

thus $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

if $x \in (A\cap B)\cup(A\cap C)$ then $x\in A\cap B$ or $x \in A\cap C$

so $x \in A$ and $B$ or $C$.

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~~2.12~~ **Thm** A countable union of countable sets is countable.

proof: $S = \bigcup_{\alpha\in A} E_\alpha$, $A$ ctble

so I can ~~write the~~ denumerate the elements of $A$ by a sequence $(\alpha_j)_{j\in\mathbb{N}}$.

$$S = \bigcup_{j\in\mathbb{N}} E_{\alpha_j}$$

might as well just write

$$S = \bigcup_{j\geq 1}^{\infty} E_j$$

now each $E_j$ can be denumerated by a sequence $x_{jk}$ $k\in\mathbb{N}$.

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \end{array}$$

denumerate $S$ in this way

now we have a mapping of
$\mathbb{N}$ onto $S$, may not be injective

but there is a subset $T \subseteq \mathbb{N}$ s.t.

$S \sim T$

(Just skip the $n$ which result in an $x_{jk}$ which has been seen already)

Thus $S$ is at most countable, and it is infinite since each $E_j$ is infinite $\square$

<u>Thm</u>: let $A$ be countable, and let $B_n$ be the set of all $n$-tuples $(a_1, \ldots, a_n)$ where each $a_k \in A$ (not necessarily distinct).

Then $B_n$ is countable.

proof: $B_1 = A$ so it is countable

suppose $B_{n-1}$ is countable. The elements of $B_n$ are of the form $(b, a)$ w/ $b \in B_{n-1}$, $a \in A$.

fix $b$. $\{(b,a) : a \in A\} \sim A$

so it is countable

and $B_n = \bigcup_{b \in B_{n-1}} \{(b,a) : a \in A\}$

countable union of countable is countable.

Cor The set $\mathbb{Q}$ of rationals is countable.

proof Apply previous thm in the case $n=2$

$B_2 = \{(a,b) : a, b \in \mathbb{Z}\}$

~~define $f : B_2 \to \mathbb{Q}$ by $f(a,b) = \frac{a}{b}$~~

~~this is surjective~~

Call $B_2' = \{(a,b) \in B_2 : a, b \neq 0$ and $a,b$ have no common divisors$\}$

$B_2' \subseteq B_2$ and hence is countable.

define $f : B_2' \to \mathbb{Q} \setminus \{0\}$

by $f(a,b) = \frac{a}{b}$, $f$ is 1-1 and onto

so $\mathbb{Q} \setminus \{0\}$ is countable.

Not all infinite sets are countable

**Thm** Let $A$ be the set of sequences of elements of $\{0,1\}$. $A$ is uncountable.

(i.e. Binary strings)

**proof**: Suppose that $A$ is countable.

then it can be denumerated by

$$s_1, \dots, s_n, \dots \in A$$

$$\begin{matrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & \cdots \end{matrix}$$

defined a binary string by

$$t(k) = 1 - s_k(k) \quad \begin{pmatrix} \text{i.e.} & = 0 & \text{if } s_k(k) = 1 \\ & = 1 & \text{if } s_k(k) = 0 \end{pmatrix}$$

now $t \neq s_n$ for any $n$ since

$$t(n) = 1 - s_n(n) \neq s_n(n),$$

this contradicts that $(s_n)_{n \in \mathbb{N}}$ denumerates $A$.

$\square$