Continuity

In this section we will consider ~~functions~~ functions

$$f : X \to Y, \quad (X, d_X) \text{ and } (Y, d_Y)$$

are metric spaces

~~We say f is cts~~

Suppose $E \subset X$ and ~~and p is a~~

$$f : E \to Y \quad \text{and} \quad p \text{ is a limit pt of } E$$

we write $f(x) \to q$ as $x \to p$ or

$$\lim_{E \ni x \to p} f(x) = q \quad \text{if}$$

$$\forall \; \varepsilon > 0 \quad \exists \; \delta > 0 \quad \text{s.t.} \quad 0 < d(x,p) < \delta$$

$$\Rightarrow \quad d_Y(f(x), q) < \varepsilon .$$

Thm let X, Y, E, p, q as before.

$$\lim_{x \to p} f(x) = q \iff \lim_{n \to \infty} f(x_n) = q$$

$\forall \{x_n\} \subseteq E$ s.t.

$x_n \neq p$ and $x_n \to p$ as $n \to \infty$.

proof: suppose $\lim_{x \to p} f(x) = q$

and let $x_n \to p$ x_n in $E \setminus \{p\}$

let $\varepsilon > 0$ $\exists$ δ s.t. $d_X(x, p) < \delta$

$\Rightarrow d_Y(f(x), q) < \varepsilon$

$x_n \to p \Rightarrow \exists N$ s.t. $n \geq N \Rightarrow d_X(x_n, p) < \delta$

$\Rightarrow d_X(f(x_n), q) < \varepsilon$.

Suppose ~~$\lim_{n \to \infty} f(x_n) = q$ for all $x_n \in E \to p$~~ $\lim_{x \to p} f(x) \neq q$

~~then~~ and then $\exists \varepsilon > 0$ ~~$\delta_n \to 0$ and~~ s.t. $\forall \delta > 0$

$\exists x_n$ w/ $0 < d(x_n, p) < \delta_n$ s.t. $d_Y(f(x), q) > \varepsilon$

take x_n as such w/ $d(x_n, p) < \frac{1}{n}$ $\square$

Cor If f has a limit at p it is unique

Thm Spse $E \subset X$ metric space, $f: E \to \mathbb{C}$, $p \in E'$

and $\lim_{x \to p} f(x) = A$ $\lim_{x \to p} g(x) = B$

then (a) $\lim_{x \to p} (f+g)(x) = A+B$

(b) $\lim_{x \to p} (f \cdot g)(x) = AB$

(c) $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ if $B \neq 0$

similar statement for $g, f: E \to \mathbb{R}^k$

$$\lim_{x \to p} (f \cdot g)(x) = A \cdot B$$

Continuous fcns

X, Y metric spaces $E \subset X$, $p \in E$

$f: E \to Y$ then we say f is cts at p

if $\forall \ \varepsilon > 0 \quad \exists \ \delta > 0$ s.t.

$$d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$$

if f is cts at every $p \in E$

then f is <u>continuous on E</u>

<u>Note:</u> if $p \in E$ is isolated then every $f : E \to Y$ is cts at p.

Typically the natural domains for continuity to be a relevant notion are perfect.

<u>Thm</u> If p is a limit pt of E then

f cts at $p \iff \lim_{x \to p} f(x) = f(p)$.

Composition

$f : X \to Y \qquad g : Y \to Z$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$g \circ f(x) := g(f(x))$$

Thm Spse X, Y, Z metric spaces, $E \subset X$

$f : E \to Y$ and $g : f(E) \to Z$ then

and define $h = (g \circ f)(x) = g(f(x))$

then if f is cts at p and

g is cts at $f(p)$

then h is cts at p.

proof: let $\varepsilon > 0$ g cts at $f(p)$ $\Rightarrow \exists\ \eta > 0$

s.t. $d(q, f(p)) < \eta \Rightarrow d(g(q), h(p)) < \varepsilon$

f cts at p $\Rightarrow$ $\exists$ $\delta > 0$ s.t.

$$d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \eta$$

$$\Rightarrow d(g(f(x)), g(f(p))) < \varepsilon$$

i.e. $d(h(x), h(p)) < \varepsilon$ Ω

Thm $f : X \to Y$ is cts iff $f^{-1}(V)$ is open for every open V in Y.

proof: We have to show that every point of $f^{-1}(V)$ is interior pt of $f^{-1}(V)$.

Suppose $p \in X$ w/ $f(p) \in V$. Since V open

$\Rightarrow$ $\exists$ $\varepsilon > 0$ s.th. $B_Y(f(p), \varepsilon) \subset V$

f cts $\Rightarrow$ $\exists$ $\delta > 0$ s.th. $d_X(x, p) < \delta$

implies $d_Y(f(x), f(p)) < \varepsilon$
i.e. $f(x) \in V$ Ω

Conversely suppose $f^{-1}(V)$ is open $\forall$ V open in Y

then let $p \in X$ and $\varepsilon > 0$ call $V = B_Y(f(p), \varepsilon)$

since $f^{-1}(V)$ open ~~$\Rightarrow \exists$~~ and $p \in f^{-1}(V)$

$\Rightarrow$ $\exists$ $\delta > 0$ s.t. $d_X(x,p) < \delta$

implies $x \in f^{-1}(V)$

i.e. $f(x) \in B_Y(f(p), \varepsilon)$

or $d_Y(f(x), f(p)) < \varepsilon$ $\square$

Cor $f: X \to Y$ is cts iff $f^{-1}(C)$ is closed for every $C \subset Y$ closed.

Proof: $f^{-1}(E^c) = [f^{-1}(E)]^c$ open

$\Rightarrow f^{-1}(E)$ closed

Thm $f, g : X \to \mathbb{C}$ cts then $f+g$, fg, f/g ($g \neq 0$ guaranteed) are cts on X

Thm (a) $f_1, \dots f_k$ real valued fcns on X

$f : X \to \mathbb{R}^k$ defined by $f(x) = (f_1(x), \dots, f_k(x))$

(f_i are called the component fcns of f)

then f is cts iff $f_1, \dots, f_k$ all cts

(b) f, g as above cts then $f+g$, $\underbrace{f \cdot g}_{: X \to \mathbb{R}}$ are cts

proof $|f_j(x) - f_j(x)| \leq \|f(x) - f(y)\| = \left(\sum_{i=1}^{k} (f_i(x) - f_i(y))^2\right)^{1/2}$

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example $x_1, \dots, x_k$ coordinates of $x \in \mathbb{R}^k$

$\phi_i(x) = x_i$ is cts fcn

since $|\phi_i(x) - \phi_i(y)| \leq |x_i - y_i| \leq |x - y|$

applying the thm repeatedly every

$x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ monomial is cts then every polynomial

$$P(x) = \sum' c_{n_1 \cdots n_k} x_1^{n_1} \cdots x_k^{n_k}$$ finite sum

is cts.

rational fcns $\dfrac{P(x)}{Q(x)}$ P, Q polynomials

are cts where $Q(x) \neq 0$.

$x \mapsto |x|$ is cts since

$$\big||x| - |y|\big| \leq |x-y| .$$

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function called Lipschitz cts if

$$d_Y(f(p), f(q)) \leq L\, d_X(p,q)$$ for some $L > 0$

function called Hölder cts if $\exists$ $\alpha \in (0,1)$
$L > 0$ s.t.

$$d_Y(f(p), f(q)) \leq L\, d_X(p,q)^{\alpha}$$

easy to see that these properties indeed imply continuity.

Continuity & Compactness

Def: A mapping $f: E \to Y$ is called bdd if $f(E)$ is a bdd set of Y.

Thm If f is a cts mapping of a compact metric space X into a metric space Y then $f(X)$ is cpct.

proof: Let $\{V_\alpha\}$ be an open cover of $f(X)$

then $f^{-1}(V_\alpha)$ open $\forall \alpha$

and since $f(X) \subseteq \bigcup_\alpha V_\alpha$

$$X \subseteq \bigcup_\alpha f^{-1}(V_\alpha)$$

so $\{f^{-1}(V_\alpha)\}$ open cover of X

X cpct $\Rightarrow \exists V_{\alpha_1}, \ldots, V_{\alpha_n}$

$$X \subset \bigcup_{k=1}^{n} f^{-1}(V_{\alpha_k})$$

but then $f(X) \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. $\square$

Thm If f is a cts mapping of a cpct space X into $\mathbb{R}^k$ then $f(X)$ is closed and bdd, in particular f bdd

Thm Suppose $f : X \to \mathbb{R}$, X cpct and

$$M = \sup_{p \in X} f(p) \qquad m = \inf_{p \in X} f(p)$$

Then $\exists$ $p, q \in X$ s.t. $f(p) = M$, $f(q) = m$.

proof $f(X)$ closed and bdd so

$$\sup f(X), \inf f(X) \in f(X) \quad \square$$

I.e. cts fcns attain their minima/maxima on compact sets.

Thm Suppose $f: X \to Y$ is cts ~~and~~ 1-1 and onto

then, X cpct, $f^{-1}: Y \to X$ defined by

$$f^{-1}(f(x)) = x \quad (x \in X)$$

is a cts mapping of Y onto X.

proof it suffices to show $f(V)$ open in Y $\forall\, V \subset X$ open

Fix $V \subset X$ open, V^c is closed in X

and hence cpct so $f(V^c)$

is cpct in Y and hence closed

Then since f is 1-1 ~~and onto~~

$$f(V) \subset f(V^c)^c$$

f is onto $\Rightarrow$ $f(V) \supset f(V^c)^c$

so $f(V) = f(V^c)^c \Rightarrow f(V)$ open

$\square$

Def. Let $f: X \to Y$ metric spaces. We say

f is uniformly cts on X if $\forall\, \varepsilon > 0$

$\exists\, \delta > 0$ s.t. $\forall\, p, q \in X$ s.t.

$$d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$$

Note: Continuity at every pt of X $\Rightarrow$

$\forall$ point p and every $\varepsilon > 0$ $\exists\, \delta$

but δ depends on both p and ε

in uniform ~~at~~ continuity ~~there is~~ given ε

there is one $\delta > 0$ for all $p \in X$.

Thm: Let f be a continuous mapping of a compact metric space X into metric space Y. Then f is u.c. on X.

proof: Let $\varepsilon > 0$ fixed. Since f is cts

$\exists$ for each $p \in X$ a number $\delta(p) > 0$

s.t. $d_X(p,q) < \delta(p) \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$

Call $G_p = B_X(p, \delta(p)/2)$

$\{G_p\}_{p \in X}$ is an open cover of X so

it has a finite subcover, $G_{p_1}, \dots, G_{p_n}$

Call $\delta = \frac{1}{2} \min\{\delta(p_1), \dots, \delta(p_n)\} > 0$

Let $p, q \in X$ s.t. $d(p, q) < \delta$

$\exists$ $m \in \{1, \dots, n\}$ s.t.

$$d_X(p, p_m) < \frac{1}{2}\delta(p_m)$$

and so $d_X(q, p_m) < d_X(q, p) + d_X(p, p_m) < \delta(p_m)$

$\Rightarrow$ $d_Y(f(p), f(p_m)) < \frac{\varepsilon}{2}$

and $d_Y(f(q), f(p_m)) < \frac{\varepsilon}{2}$

So $d_Y(f(p), f(q)) < d_Y(f(p), f(p_m))$
$+ d_Y(f(q), f(p_m))$

$< \varepsilon$ $\square$

Examples

let $E \subset \mathbb{R}$ suppose E is not closed

so that $\exists\ x_0 \in \mathbb{R} \setminus E$ which is a limit pt of E

define $f(x) = \dfrac{1}{x - x_0}$. This is cts on

E but not bounded and not u.c.

(given $\varepsilon > 0$, and $\delta > 0$ arbitrary

choose $x \in E$ s.t. $|x - x_0| < \delta$

if t close enough to x_0 we

can make $|f(t)| > M + |f(x)|$ for any $M > 0$

so $|f(t) - f(x)| \geq |f(t)| - |f(x)|$

$> M$

Also we can make a function s.t. the maximum

is not achieved as

see $f(x) = \dfrac{1}{1 + (x - x_0)^2}$

$\sup_E f = 1$ but $f < 1$ on E.

if E is unbounded we can also make counterexamples.

Continuity and Connectedness

Thm If $f: X \to Y$ cts and $E \subset X$ is connected

then $f(E)$ is connected.

proof: Suppose that $\exists\ U, V \subset Y$ disjoint

s.t. $$f(E) \subset U \cup V$$

and $$f(E) \cap U \neq \phi, \quad f(E) \cap V \neq \phi.$$

Call $A = f^{-1}(U)$, $B = f^{-1}(V)$

open b/c inverse image of open set is open for f cts.

then $A \cap E \neq \phi$, $B \cap E \neq \phi$

and $A \cap B = \phi$ so E not connected ⊠

Thm Let f cts r.v. on interval $[a,b]$.

If $f(a) < f(b)$ ~~then~~ and c s.t.

$f(a) < c < f(b)$ then $\exists\, x \in (a,b)$ s.t.

$f(x) = c$.

proof: $[a,b]$ is connected so

$f([a,b])$ is connected so it must

contain $[f(a), f(b)]$.