Recall from last time:

We showed that for (X,d) cpct metric space

$$C(X) = \{ f : X \to \mathbb{R} : f \text{ cts} \}$$

w/ $\|f\|_{sup} = \sup_{x \in X} |f(x)|$

is a Banach Space, but we needed the following theorem:

Thm: Suppose $f_n : X \to \mathbb{R}$ are cts and $f : X \to \mathbb{R}$ some function. If $\|f_n - f\|_{sup} \to 0$ as $n \to \infty$ then f is cts as well.

proof: let $x \in X$, $\varepsilon > 0$. $\exists$ N s.t. $n \geq N$

$$\Rightarrow \|f_n - f\|_{sup} < \frac{\varepsilon}{3}$$

f_N cts $\Rightarrow$ $\exists$ $\delta > 0$ s.t. $d(x,y) < \delta$

$$\Rightarrow |f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$$

Then ~~$\forall y$~~ $d(y,x) < \delta \Rightarrow$

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\square$

Compactness Revisited

A ~~set~~ metric space (X,d) is called **Totally Bounded**

if for all $\varepsilon > 0$ $\exists$ $x_1, \dots, x_n \in X$

s.t. $$X = \bigcup_{i=1}^{n} B(x_i, \varepsilon).$$

Thm For metric space (X,d) TFAE

(i) X compact

(ii) X sequentially compact

(iii) X complete and totally bounded.

I'm having you prove in the homework that

sequentially compact $\Rightarrow$ complete and totally bdd.

We already showed

compact $\Rightarrow$ sequentially compact

Let's show that Complete & totally bdd

$\Rightarrow$ compact.

First let's note that any subset of a totally bdd metric space ~~set~~ ~~is totally bdd~~ is totally bdd

~~for $\varepsilon > 0$~~

~~Can be covered by a finitely many balls~~

let $E \subset X$, $\varepsilon > 0$.

X totally bdd $\Rightarrow$ $E \subset X = \bigcup_{i=1}^{n} B(x_i, \varepsilon)$

so $E = \bigcup_{i=1}^{n} B(x_i, \varepsilon) \cap E$.

Suppose X is not cpct so $\exists$ an open cover $\{G_\alpha\}$ w/ no finite subcover.

Since X is totally bdd we can cover it by balls $B(x_1, 1/2), \ldots, B(x_n, 1/2)$ of diameter (at most) 1.

Since there are only finitely many such balls $\{G_\alpha\}$ must not have a finite subcover for at least one of them.

Call that set E_1.

Then apply the argument ~~again~~ inductively.

given E_k ~~define~~ cover E_k by finitely many balls $B(x_j, \frac{1}{2(k+1)}) \cap E_k$ $j=1,\dots,n$

$\{G_\alpha\}$ must not have a finite subcover for one of these, call it E_{k+1}

in this way we get nested sets E_k of diameter $\leq \frac{1}{k}$ such that $\{G_\alpha\}$ has no finite subcover for E_k

In particular each $E_k \neq \phi$.

let $x_k \in E_k$, we claim x_k is Cauchy. let $\varepsilon > 0$ and $N \geq \frac{1}{\varepsilon}$

then for $n, m \geq N$, ~~$x_n, x_m \in$~~

$$x_n \in E_n \subseteq E_N, \quad x_m \in E_m \subseteq E_N$$

$$d(x_n, x_m) \leq \operatorname{diam} E_N \leq \frac{1}{N} \leq \varepsilon.$$

So $\{x_n\}_{n=1}^{\infty}$ is Cauchy in X and hence it converges to some x_∞ (since X complete)

Since $\{G_\alpha\}$ open cover of X $x_\infty \in G_{\alpha_0}$ for some α_0 and $\exists\, r > 0$ st.

$$B(x_\infty, r) \subset G_{\alpha_0}.$$

Let $N \geq \frac{1}{r}$ so then for all $x \in E_N$

(since $x_\infty \in \overline{E_n}$ for all n and $\operatorname{diam} \overline{E_n} = \operatorname{diam} E_n$)

$$d(x_\infty, x) \leq \operatorname{diam} \overline{E_N} = \operatorname{diam} E_N \leq \frac{1}{N} < r$$

so

$$E_N \subset B(x_\infty, r) \subseteq G_{\alpha_0}$$

Which is a contradiction of the choice of E_N. $\square$

Sequential Compactness (Revisited)

Recall that a metric space (X, d) is called <u>sequentially compact</u> if every sequence in X has a convergent subsequence.

~~We~~ Let's reprove some of our main thms using the sequential compactness defn (which we now know is equivalent to open cover compactness) just to see how the defn is used in practice.

Thm let E closed and K cpct disjoint subsets of metric space (X, d).

Then $$\inf_{x \in E,\, y \in K} d(x,y) > 0 .$$

proof: Suppose otherwise then $\forall\, n > 0$ $\exists$ $x_n \in E$, $y_n \in K$ s.t.

$d(x_n, y_n) < \frac{1}{n}$ ($\frac{1}{n}$ not a LB for $\{d(x,y) : x \in E, y \in K\}$)

$y_n \in K$ so has a convergent subseq $y_{n_k} \to y \in K$.

Then we claim y is a limit pt of E

$\Rightarrow$ $y \in E \cap K$ which is a contradiction

let $r > 0$, $\exists$ N s.t. for $k \geq N$

$N \geq \frac{2}{r}$ s.t. $d(x_{n_k}, y_{n_k}) \leq \frac{r}{2}$ and $d(y_{n_k}, y) \leq \frac{r}{2}$

so $d(x_{n_k}, y) \leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, y)$

$\leq r$

$\square$

<u>Thm</u> Suppose f is a cts map of a cpct metric space X into a metric space Y. Then $f(X)$ is cpct.

proof: let y_n be a sequence in $f(X)$

i.e. $y_n = f(x_n)$ for some $x_n \in X$

X cpct $\Rightarrow$ $\exists$ $x_{n_k} \to x \in X$

f is cts $\Rightarrow$ $f(x_{n_k}) \to f(x)$ as $k \to \infty$

i.e. $y_{n_k} \to f(x) \in f(X)$ $\square$

Thm let f be a ~~uniformly~~ continuous map of a cpct metric space (X, d_X) into a metric space (Y, d_Y).
Then f is uniformly cts.

proof: Suppose otherwise. Then $\exists\ \varepsilon > 0$ s.t. $\forall\ n$

~~~~

there are ~~~~ $p_n, q_n \in X$ w/

$$d_X(p_n, q_n) < \frac{1}{n} \quad \text{but} \quad d_Y(f(p_n), f(q_n)) > \varepsilon.$$

Since X cpct $\exists$ subsequence of p_n

$$p_{n_k} \to p$$

First lets see that $q_{n_k} \to p$ as well

let $\varepsilon > 0$ and $k \geq \frac{2}{\varepsilon}$ (so that $n_k \geq \frac{2}{\varepsilon}$ as well) so that $d(p_{n_k}, p) < \varepsilon/2$

then $d(q_{n_k}, p) \leq d(q_{n_k}, p_{n_k}) + d(p_{n_k}, p)$

$$\leq \frac{1}{n_k} + \frac{\varepsilon}{2} \leq \varepsilon.$$
~~~~

so ~~$f(p) = \lim$~~ since f is cts at p

$$\lim_{k\to\infty} f(q_{n_k}) = f(p) = \lim_{k\to\infty} f(p_{n_k})$$

$$\Rightarrow \quad 0 = \lim_{k\to\infty} d_Y(f(q_{n_k}), f(p_{n_k})) \geq \varepsilon$$

Which is a contradiction $\square$