Compact Sets

This is an extremely important notion in analysis. The first definition we give may be a bit unintuitive.

We say that $\{G_\alpha\}$ is an open cover

of a set $E \subset X$, (X, d_X) metric space,

if each G_α open and $E \subset \bigcup_\alpha G_\alpha$.

We say that $K \subset$ metric space X is

compact if every open cover contains

a finite subcover.

i.e. if $\{G_\alpha\}$ is an open cover of K

then $\exists$ $\alpha_1, \dots, \alpha_n$ s.t.

$$K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}.$$

Compactness, unlike open/closed, behaves well w.r.t. relative-ness

Thm Spse $K \subset Y \subset X$. Then K cpct rel. to Y $\Leftrightarrow$ K cpct rel to X.

This ~~justifies~~ means that talking about a compact metric space makes sense

proof: I'll skip the proof read Thm's 2.30 and 2.33 in the book

Thm: Cpct sets are closed.

proof: Let K cpct $\subset X$ metric space.

We will show K^c open, if $K^c = \emptyset \Rightarrow$ done

let $p \in K^c$. For each $q \in K$ $\exists$ $d(p,q) > 0$

~~so $B\left(p, \frac{d(p,q)}{2}\right)$~~

Call $V_q = B(p, \frac{d(p,q)}{2})$

$W_q = B(q, \frac{d(q,p)}{2})$

which are disjoint

$\{W_q\}_{q \in K}$ covers K (since $q \in W_q$)

so $\exists$ $q_1, \dots, q_n$ s.t. $\{W_{q_i}\}_{i=1}^{n}$ covers K.

Let $V = V_{q_1} \cap \dots \cap V_{q_n}$ which is open

$p \in V$ and

$\cancel{V \cap K}$ $V \subset \bigcap_{i=1}^{n} W_{q_i}^c \subset K^c$

so p is an interior pt of K^c.

$\square$

Thm: Closed subsets of compact sets are cpct

proof: Let E closed $\subset K$ cpct

Let $\{G_\alpha\}$ be an open cover of E.

$\{G_\alpha\} \cup \{E^c\}$ is an open cover of K.

$\Rightarrow \exists$ $G_1, \dots, G_n$ open cover of E

(E^c does not need to be in there since $E^c \cap E = \phi$)

Thm If $\{K_\alpha\}_{\alpha \in A}$ is a collection of compact sets of metric space (X, d) such that for every finite subcollection $I \subset A$ $\bigcap_{\alpha \in I} K_\alpha \neq \phi$, then

$$\bigcap_{\alpha \in A} K_\alpha \neq \phi.$$

proof: for each K_α put $G_\alpha = K_\alpha^c$.

Suppose that $\bigcap_{\alpha \in A} K_\alpha$ is empty

~~then~~ ~~$\exists$~~ then for each $\beta \in A$ fixed

$$K_\beta \subset \bigcup_{\alpha \in A} G_\alpha$$

$\Rightarrow$ $\exists$ a finite subcover $G_{\alpha_1}, \ldots, G_{\alpha_n}$ of K_β

but then $K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \cap K_\beta = \phi$

10/13 contradicting the finite intersection property.

Thm (Compact $\Rightarrow$ sequentially compact)

If E is an infinite subset of a cpt set K then ~~E~~ E has a limit point in K.

proof: Suppose otherwise. Then for each $p \in K$ $\exists$ a nbhd $B(p, r_p)$ s.t. $B(p, r_p) \cap E$ has at most one element

$\{B(p, r_p)\}_{p \in K}$ covers K and so it has a finite subcover and

$$\underbrace{E \subset K \subset \bigcup_{i=1}^{n} B(p_i, r_{p_i})}$$

has at most n elements of E

This contradicts E infinite. $\square$

Now we begin on a proof of the following important theorem which shows that several definition of compactness are equivalent

<u>Thm</u>: let $E \subseteq \mathbb{R}^n$ w/ Euclidean metric
TFAE

(i) E closed and bdd
(ii) E compact
(iii) every infinite subset of E has a limit point in E.

<u>Remark</u>: (ii) and (iii) are equivalent in <u>any</u> metric space, but (i) in general is not.

We will just prove the theorem in $\mathbb{R}$.
read the book for proof in $\mathbb{R}^n$.

First let's show that closed intervals

$[a,b] \subseteq \mathbb{R}$ are compact.

~~Thm (FIP for intervals) If I_n is~~

Thm (Finite intersection Property for intervals)

If I_n is a nested ($I_{n+1} \subseteq I_n$)

sequence of bdd sub-intervals of $\mathbb{R}$ then

$\bigcap_{n=1}^{\infty} I_n$ is not empty.

proof: $I_n = [a_n, b_n]$

Let $E = \{a_n : n = 1, 2, \ldots\}$

$a_n \le b_p$ for all n so E bdd above

Call $x = \sup E$.

Then $a_n \le x$ for all n.

on the other hand

$a_n \le a_{n+m} \le b_{n+m} \le b_m$

so b_m are all UB for E

so $x \le b_m$ for all m

$\Rightarrow$ $x \in I_n$ $\forall$ n $\square$

<u>Thm</u> Closed bounded intervals $I = [a,b]$ are compact.

<u>proof</u>: suppose there is an open cover $\{G_\alpha\}$ of I w/ no finite subcover.

divide I into half

$$[a,b] = [a, \tfrac{a+b}{2}] \cup [\tfrac{a+b}{2}, b]$$

$\{G_\alpha\}$ covers left and right halfs and must fail to have a finite subcover for one of the two

Call that interval I_1.

Apply the same reasoning inductively to get

I_j $j \in \mathbb{N}$ nested intervals

w/ $(b_j - a_j) = 2^{-j}(b-a)$

by the NIP of intervals

$\bigcap_{j=1}^{\infty} I_j$ is nonempty, contains some x_*

~~since G.~~ let α s.t. $x_* \in G_\alpha$

since G_α open $\exists\, r > 0$ s.t.

$B(x_*, r) \subseteq G_\alpha$.

let j sufficiently large (by archimedean property)

s.t. $2^{-j}(b-a) < \frac{r}{2}\, r$

then $x_* \in I_j$

so $b - x_* \leq 2^{-j}(b-a) < r$

$x_* - a \leq 2^{-j}(b-a) < r$

so $I_j \subseteq B(x_*, r) \subseteq G_\alpha$

this is a finite ~~cover~~ subcover of I_j

$\rightarrow\leftarrow$

Theorem (Heine-Borel)

~~For~~ set $E \subseteq \mathbb{R}$ (or $\mathbb{R}^k$) ~~then~~ TFAE

(a) E is closed & bdd
(b) E is compact
(c) Every infinite subset of E has a limit point in E.

Rem! (b) $\Leftrightarrow$ (c) true in general metric space, but harder

proof:

If (a) holds then bdd $\Rightarrow$

$E \subset [a,b]$ for some a, b

so closed subset of cpct set is cpct

$\Rightarrow$ (b)

(b) $\Rightarrow$ closed we know,

also E bdd since for any fixed $x \in X$

$\{B(x,R)\}_{R>0}$ is an open cover of E.

take finite subcover $\{B(x,R_j)\}_{j=1}^{n}$

and let $R_* = \max_{j=1,\dots,n} R_j$.

then $E \subset B(x, R_*)$.

so (b) $\Rightarrow$ (a).

~~Now lets show (c) $\Rightarrow$ (b)~~
~~(we already know (b) $\Rightarrow$ (c))~~

Finally we show (c) $\Rightarrow$ (a)

Supposed E not bdd

$\Rightarrow \exists$ $x_n \in E$ w/ $|x_n| > n$

$\{x_1, x_2, \ldots\}$ is infinite

and has no limit points

so E is bdd

Suppose E is not closed $\Rightarrow \exists$ $x_* \in \mathbb{R}$ limit pt of E but not in E.

Since x_* is a limit pt of E

$\forall$ $n \in \mathbb{N}_+$ $\exists$ $x_n \in E$ w/

$$|x_n - x_*| < \frac{1}{n} \quad (\text{and } x_n \neq x_*)$$

let $S = \{x_1, x_2, \ldots\}$

then S is infinite

S has x_* as a limit point and ~~not~~ no other $y \in \mathbb{R}$ since

$$|x_n - y| \geq |y - x_*| - |x_n - x_*| > |y - x_*| - \frac{1}{n}$$

which is $> \frac{1}{2}|x_* - y|$

for all but finitely many n

$\Rightarrow$ y not a limit point of E if $y \neq x_*$

thus since S has a limit pt in E

that must be x_* so $x_* \in E$ $\rightarrow\leftarrow$

Perfect Sets

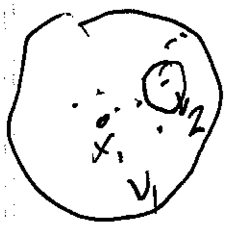
Thm let P be a non-empty perfect set in $\mathbb{R}^n$. Then P is uncountable.

proof: Since P has limit pts P must be infinite. Suppose P countable.

enumerate P by $x_1, x_2, \ldots$

Call $V_1 = B(x_1, r)$ ~~for any~~ some $r > 0$

$$\bar{V}_1 = \{x : |x - x_1| \leq r\}$$



suppose V_n constructed and

$V_n \cap P$ non-empty

since every pt of P is a limit pt of P

we can choose a nbhd V_{n+1} s.t.

$\overline{V_{n+1}} \subset V_n$, $x_n \notin \overline{V_{n+1}}$ and $V_{n+1} \cap P \neq \phi$

construct V_n inductively in this way

Call $K_n = \overline{V_n} \cap P$ closed, bdd and hence cpct sets of $\mathbb{R}^n$

$x_n \notin K_{n+1}$ ~~Wh~~ so $\displaystyle\bigcap_{n=1}^{\infty} K_n \cap P = \phi$

since $K_n \subset P \Rightarrow \displaystyle\bigcap_{n=1}^{\infty} K_n = \phi$

but ~~by~~ since each K_n nonempty and $K_{n+1} \subset K_n$

$\displaystyle\bigcap_{n=1}^{\infty} K_n$ nonempty $\rightarrow\leftarrow$.