The Cantor Set

Recall that a perfect set E in a metric space X

is s.t.h every ~~$x \in E$~~ is a limit pt of E.

Also Recall that perfect sets are uncountable.

~~Wt CD~~

The Cantor Set is an important example

~~beca~~ for several reasons

It is a perfect subset of $\mathbb{R}$ that contains

~~the~~ no interval.

It is also (as a corollary) totally disconnected

in the sense that its only connected subsets

are points.

Much later you will see it again as

an example of a set with "dimension"

strictly between 0 and 1.

The middle ~~third~~ $\frac{1}{3}$ Cantor set is

~~the Cantor set is~~

$$C = \left\{ x \in [0,1] : x \text{ has no 2's in it's ternary expansion} \right\}$$

(although this classification isn't so useful)

the better construction is inductive

Call $C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3}, 1]$

(remove the middle ~~third~~ $\frac{1}{3}$ of the interval)

$$\cancel{C_{n} = C_{n-1} \setminus \{ x \in [0,1] : 1 < 3^n x}$$

then $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

at each stage

$$C_n = \bigcup_{i=1}^{2^n} I_i^n \qquad I_i^n \text{ closed intervals}$$

of length 3^{-n}.

$C_n \subset C_{n-1} \subset \cdots$

given C_n define C_{n+1} as ($I^n_i - [a^n_i, b^n_i]$)

$$C_{n+1} = \bigcup_{i=1}^{2^n} \cancel{[a^n_i + \frac{(b^n_i - a^n_i)}{3}]} \; [a^n_i, a^n_i + \frac{(b^n_i - a^n_i)}{3}] \cup [b^n_i - \frac{(b^n_i - a^n_i)}{3}, b^n_i]$$

Indeed this is union of $2^{(n+1)}$ closed disjoint intervals of length $\frac{1}{3} \cdot \text{length } I^n_i = 3^{(n+1)}$

Call $C = \bigcap_{n=1}^{\infty} C_n$ the <u>middle $\frac{1}{3}$ Cantor Set</u>

P is cpct and non-empty (nested intersection of cpct)

Further no segment

$\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$ intersects C

(since it is removed at stage n)

but every interval $I \subset \mathbb{R}$ has a segment of this form contained in it. $I = (\alpha, \beta)$

let n s.t. $3^{-n} \le \frac{\beta-\alpha}{6}$

Let k minimal in $\mathbb{Z}$ s.t. $\frac{3k+1}{3^n} \in (\alpha, \beta)$, by minimality

$$\frac{3(k-1)+1}{3^n} \le \alpha$$

so $\frac{3k+1}{3^n} - \alpha \le \frac{1}{3^{n-1}}$

so $\frac{3k+2}{3^n} - \alpha \le \frac{1}{3^n} + \frac{1}{3^{n-1}} = \frac{4}{3^n} \le \frac{4(\beta-\alpha)}{6} < \beta - \alpha$

To show that C is perfect we show that C has no isolated point.

Let $x \in C$ and $A = (\alpha, \beta)$ be an open interval containing x.

Since $x \in C_n$ $\forall$ n let I_n be the interval of C_n containing x.

for n suff large $(3^{-n} < \min\{x-\alpha, \beta - x\})$

$I_n \subset A$. Let x_n be ~~the other~~ an endpoint of I_n which is not x.

~~then~~ thus every open interval containing x intersects C at a pt other than x.

Hence C is perfect.

Aim: C has measure zero.