MATH 20000: Topics on Final Exam

Math 200: Section 11, Autumn 2015, Instructor: William Feldman

The test is cumulative, covering all the topics from the first and second exams and in addition the topics from Chapters 7.3-7.6 and 8.1.

- **General Advice**
  - Make your drawings very carefully, make them large enough to see the important details, clearly label the axes, clearly label several important points which help to interpret the scale.
  - Review what the domain and range of a function are if you are not clear on this. Understand the difference between a scalar valued function and a vector field. I am especially interested in checking that your basic understanding of the properties of the objects we are studying (scalars, vectors, and to a lesser degree matrices) is sound.
- **Chapter 1**
  - Vector operations (addition, subtraction, scalar multiplication) and their geometric interpretation.
  - The parametric equation of a line given (1) point and vector, (2) two points.
  - The inner product, the norm, their geometric interpretation.
  - Orthogonal projection.
  - Determinants of 2×2 and 3×3 matrices and their geometric interpretation (as area/volume of parallelogram and parallelopiped respectively).
  - Cross product, geometric interpretation, algebraic rules.
  - Triple product ($(a \times b) \cdot c$) relation with determinants of 3×3 matrices.
  - Equation of a plane in $\mathbb{R}^3$ given point and normal vector.
  - Distance from a point to a plane.
  - Definition of cylindrical and spherical coordinate systems and basic geometry, converting between coordinate systems.
  - Inner product in $\mathbb{R}^n$, matrix multiplication and matrix vector multiplication.
- **Chapter 2**
  - The definition of a vector valued function, its domain and range.
  - The graph of a function, level sets, sections. Using the level sets and sections to produce an accurate drawing of the graph.
  - Open sets, definition of limits by neighborhoods.
  - Basic properties of limits.
  - Definition of continuity, basic properties, checking whether a function is continuous.
  - The partial derivative of a function of multiple variables.
  - Definition of differentiability for vector valued functions of several variables (i.e. $f : \mathbb{R}^n \to \mathbb{R}^m$), calculating the matrix Df. What is a C^1 function on a domain $U \subset \mathbb{R}^n$.
  - The tangent plane to the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$.
  - Relationship between differentiability and continuity.
  - Paths/curves in $\mathbb{R}^n$ (what is the distinction?). Parametrizations, re-parametrizations of a curve by different paths.
  - Velocity vector of a path, speed of a path, tangent vector/line to a curve.
  - Differentiation rules, products, sums, constant multiples, quotients, and the chain rule.
  - Directional derivatives.
  - Understanding the gradient ∇f as direction of fastest increase, normal to level surfaces of f.
  - The equation of the tangent plane to a level surface of $f : \mathbb{R}^n \to \mathbb{R}$.
- **Chapter 3**
  - What is a C^2 function, the mixed partial derivatives, equality of mixed partials.
  - Taylor's Theorem up to second order for functions of several variables and its interpretation.
  - Local minima, maxima, critical points.
  - First derivative test for local extrema.
  - Quadratic functions in $\mathbb{R}^n$ and the Hessian of $f : \mathbb{R}^n \to \mathbb{R}$.

- ◦ Positive/negative definiteness for a quadratic.
  - ◦ The second derivative test for local extrema, esp. the case $n = 2$ where it is much easier to check positive-definiteness.
  - ◦ Saddle points.
  - ◦ Strategy for finding the global maxima and minima of $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ by parametrizing ∂U.
  - ◦ The method of Lagrange multipliers for constrained extrema.
  - ◦ The method of Lagrange multiplies for finding the global maxima and minima of $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$.
- **Chapter 4**
  - ◦ Velocity, acceleration of a path.
  - ◦ Differentiation rules for paths.
  - ◦ Checking whether a path solves a differential equation like Newton's Law $ma(t) = mc''(t) = F(c(t))$
  - ◦ Arc length of a path.
  - ◦ Vector fields, gradient vector fields, flow lines of vector fields (drawing the vector field, drawing the flow lines, checking whether a path c is a flow line of a vector field V).
  - ◦ Divergence of a vector field, curl of a vector field. How to compute them, what kind of functions are they (scalar, vector), basic facts like gradients have $\nabla \times (\nabla f) = 0$ for f scalar valued function on $\mathbb{R}^3$ and $\nabla \cdot (\nabla \times F) = 0$ for $F : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field.
- **Chapter 5**
  - ◦ Cavalieri's principle, reducing integral over a rectangle to iterated integrals.
  - ◦ Fubini's theorem (being able to apply it).
  - ◦ Computing integrals over elementary regions in $\mathbb{R}^2$ and $\mathbb{R}^3$, changing the order of integration. You should be able to interpret in all directions between a description of the domain in words (e.g. the region of $x^2 + y^2 + z^2 \leq 1$ with $z \geq 0$), an iterated integral over the domain, a description of the domain as an elementary domain in set theoretic notation.
  - ◦ Tricks to compute integrals over non-elementary regions like splitting up into several elementary regions and summing.
- **Chapter 6**
  - ◦ Definitions of one-to-one, onto and invertible. Definition of the image of a set under a mapping.
  - ◦ Images of linear maps (parallelograms are mapped to parallelograms), computed the area of a parallelogram (or parallelopiped) by the determinant of a matrix/linear map.
  - ◦ The jacobian determinant of a transformation.
  - ◦ The change of variables theorem, computing integrals using linear changes of variables, polar, cylindrical and spherical coordinates.
  - ◦ Expressing domains of $\mathbb{R}^3$ in different coordinate systems (closely related with computing integrals using change of variables).
- **Chapter 7**
  - ◦ The path integral, the line integral (what is the difference with path integral?). Computing these in various examples.
  - ◦ Reparametrizations of paths and how the line/path integral can change under reparametrization.
  - ◦ Line integrals of gradient vector fields.
  - ◦ Line integrals over simple curves and over curves with several components.
  - ◦ Parametrizations of surfaces, general concepts and specific common examples like parametrizing a plane, a graph of a function, a sphere, seeing when to use spherical or cylindrical coordinates to come up with parametrization.
  - ◦ Tangent vectors T_u, T_v to a parametrized surface at a point, tangent plane to a parametrized surface at a point, regular points of a surface, normal vector and unit normal vector to a surface at a point.
  - ◦ The area of a parametrized surface, the integral of scalar functions over surfaces.
  - ◦ Oriented surfaces and surface integrals of vector fields on oriented surfaces.
- **Chapter 8**

- ∘ Green's Theorem and its rephrasing as Stokes' Theorem and the Divergence Theorem in $\mathbb{R}^2$. Applying this in a simple domain or in a domain with holes (making sure the line integrals on the boundary are oriented correctly, counter-clockwise for outer boundary clockwise for inner boundaries).