MATH 6410: Ordinary Differential Equations

Instructor: Will Feldman

University of Utah

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Course Information

Course information

Instructor: William Feldman (he/him/his)

Office: JWB 101

Webpage: `math.utah.edu/~feldman/teaching.html`

Canvas: Canvas page will be used for grades, Zoom info, any possible lecture recordings.

Office Hours: TBA, will be held in Zoom room listed above.

TA: Keshav Patel

COVID note: Be prepared for possible Zoom classes.

Course information

Primary Textbooks:

Differential Equations and Dynamical Systems, Lawrence Perko.

Ordinary Differential Equations and Dynamical Systems, Gerald Teschl [pdf].

Supplementary Texts:

Theory of Ordinary Differential Equations, Christopher P. Grant.

Additional course materials will be shared online.

Grading policy

Course work:

- **50%** - Homework - Individual problems assigned approximately 3-5 per week depending on length, typically due 7 days after they are assigned. Scores of lowest 15% of problems will be dropped at end of semester.
- **50%** - Final Exam - The final exam is on Monday, December 13, 2021 at 1:00 – 3:00 pm.

Your final letter grade will be determined by the following rubric:

A : 90%+

A- : 85%-90%

B+ : 80%-85%

B : 70% - 80%

C : below

Initial value problems

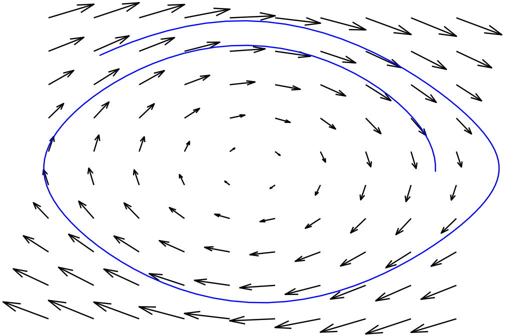
Initial value problems

Initial value problem (IVP) for a first order system: find $x(t) : [0, T] \to \mathbb{R}^n$ solving

$$\begin{cases} \dot{x}(t) = f(t, x(t)) & \text{for } t \in (0, T) \\ x(0) = x_0. \end{cases} \tag{1}$$

where $f : U \to \mathbb{R}^n$ is continuous, $U \subset \mathbb{R}^n$ open, and $x_0 \in U$.

Following a vector field



The fundamental questions

Mathematical theory of differential equations always starts with a set of questions called **well-posedness**

- **Existence**: "Is there a solution?"
  - Local (i.e. small interval around initial time)
  - Global (all positive times)
- **Uniqueness**: "Is there only one solution?"
  - Regularity of the dependence on the data.
  - Regularity of the dependence on the equation.

Qualitative features

Once we have understood the basics we can begin to ask more refined questions and start to really understand the solutions of a particular ODE.

- Long time / asymptotic behavior of solutions
- Dependence on the equation
  - Bifurcation theory
  - Perturbation theory

Higher order equations

What about differential equations involving higher order derivatives? Why just consider first order systems?

General abstract higher order ODE, look for $y(t)$ solving

$$\begin{cases} F(t, y, y^{(1)}, \dots, y^{(k)}) = 0 & \text{for } t > 0 \\ y^{(j)}(0) = y_{0,j} & \text{for } 0 \le j \le k-1. \end{cases} \tag{2}$$

If $\partial_1 F(\vec{y}_0, 0) \neq 0$ then, by implicit function theorem, can solve for $y^{(k)}$

$$y^{(k)}(t) = g(t, y(t), \dots, y^{(k-1)}(t))$$

Higher order equations (ctd...)

Now write

$$x(t) = (y(t), \dots, y^{(k-1)}(t))^T \in \mathbb{R}^k$$

Which solves the first order system

$$\begin{cases} \dot{x}(t) = f(t, x(t)) & \text{for } t > 0 \\ x(0) = (y(0), y^{(1)}(0), \dots, y^{(k-1)}(0))^T \end{cases} \tag{3}$$

with

$$f(t, x) = (x_2, x_3, \dots, x_{k-1}, g(t, x_1, \dots, x_k))^T$$

Examples and ideas

Newton's Equations

Particle of mass $m > 0$ at position $x(t)$ moves in $\mathbb{R}$ under the influence of a force field $F : \mathbb{R} \to \mathbb{R}$ and a kinetic frictional force (coefficient $\mu \geq 0$) opposing motion

$$m\ddot{x} = -\mu\dot{x} + F(x).$$

It is common to view this as a first order system by considering the equation for $(x, p) = (x, m\dot{x})$ the position and momentum

$$\begin{cases} \dot{x} = \frac{1}{m}p \\ \dot{p} = -\frac{\mu}{m}p + F(x). \end{cases} \tag{4}$$

Newton's Equations

Newton's equations have a conserved ($\mu = 0$) / dissipated ($\mu > 0$) quantity. Multiply the ODE by $\dot{x}$

$$m\ddot{x}\dot{x} - \dot{x}F(x) = -\mu\dot{x}^2$$

and note that the left hand side is a derivative

$$\frac{d}{dt}(\frac{p^2}{2m} + V(x)) = -\mu\dot{x}^2 \leq 0$$

where

$$V(x) = -\int_0^x F(u)\ du.$$

Newton's Equations

This quantity is called the **Hamiltonian**

$$H(p,x) = \frac{p^2}{2m} + V(x)$$

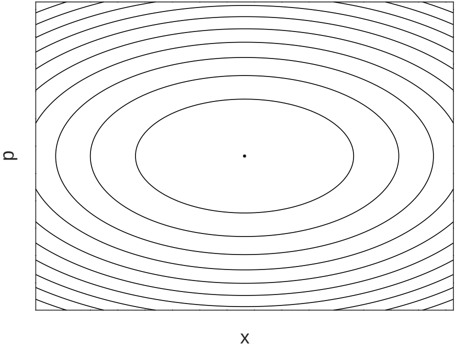
When $\mu = 0$ the solutions of Newton's equations remain on level sets of the Hamiltonian and the first order system has the **Hamiltonian form**

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p}(p,x) \\ \dot{p} = -\frac{\partial H}{\partial x}(p,x). \end{cases}$$

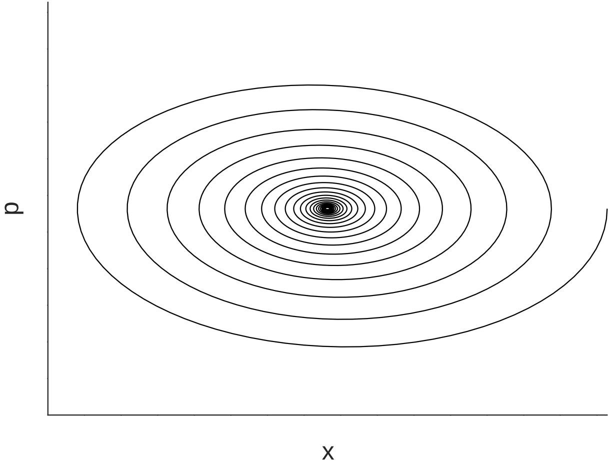
Newtons equations: mass on spring

Ideal point mass m on a spring with constant k, rest state at 0,

$$H(p,x) = \frac{p^2}{2m} + \frac{1}{2}kx^2.$$



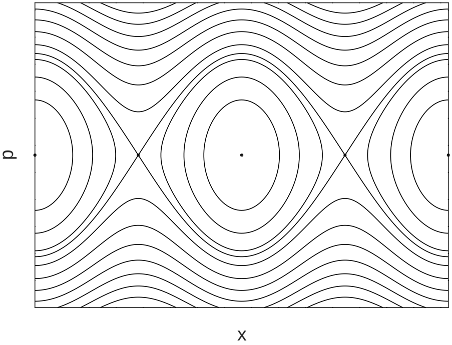
Damped mass on spring



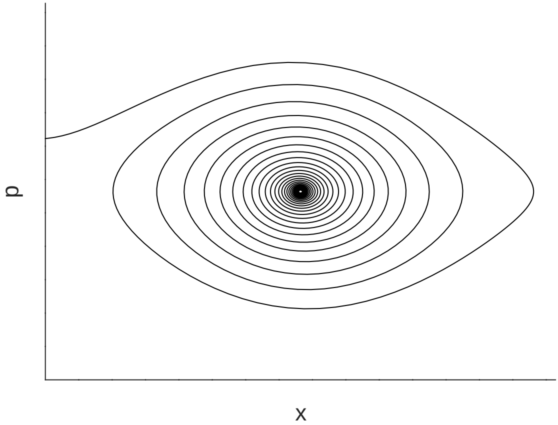
Newtons equations: pendulum

Ideal point mass m hung on a string of length L, θ is the angle from the vertical axis and $p = mL\dot{\theta}$ is the momentum

$$H(p, x) = \frac{p^2}{2m} + mgL(1 - \cos\theta).$$



Damped pendulum



Population dynamics

Most basic equation of population dynamics is the most basic ODE of all

$$\dot{N} = rN \text{ and } N(0) = N_0.$$

Models, for example, population growth of some species, nuclear decay, etc. The number r is the reproduction / decay rate.

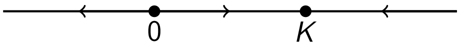
Slightly more sophisticated is the logistic growth model

$$\dot{N} = rN(1 - N/K) \text{ and } N(0) = N_0.$$

Here K is the carrying capacity.

Phase line analysis

Plot the positive/negative regions of $f(N) = rN(1 - N/K)$ on $\mathbb{R}$



Solutions converge to **critical point** (zero of f and **stationary solution** of the ODE) either 0 or K based on where initial data sits. We will make this rigorous later.

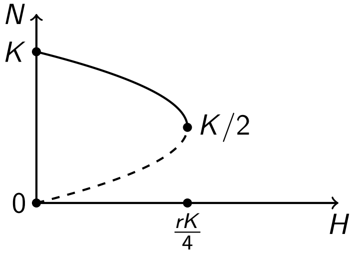
Population dynamics

Now if we add in "harvesting" to the logistic growth model

$$\dot{N} = rN(1 - N/K) - H \quad \text{and} \quad N(0) = N_0$$

where H is the number of the population harvested per unit time.

For $H < \frac{rK}{4}$ there are two positive fixed points, which collide and annihilate at $H = \frac{rK}{4}$, called a **saddle-node bifurcation**.



Chemical reaction

Reversible synthesis/decomposition reaction between chemicals A and B producing AB

$$A + B \underset{k_2}{\overset{k_1}{\rightleftharpoons}} AB$$

forward reaction occurs with rate k_1 backwards occurs with rate k_2.

Call $n_X(t)$ to be the concentration of reactant X at time t, then the densities evolve by the ODE system

$$\begin{cases} \dot{n}_{AB} = k_1 n_A n_B - k_2 n_{AB} \\ \dot{n}_A = \dot{n}_B = k_2 n_{AB} - k_1 n_A n_B. \end{cases} \tag{5}$$

Chemical reactions: homework

Problem

Consider the chemical reaction system

$$\begin{cases} \dot{n}_{AB} = k_1 n_A n_B - k_2 n_{AB} \\ \dot{n}_A = \dot{n}_B = k_2 n_{AB} - k_1 n_A n_B. \end{cases}$$

- *Show that* $n_A - n_B$, $n_{AB} + n_A$, $n_{AB} + n_B$, *and* $n_{AB} + \frac{1}{2}(n_A + n_B)$ *are all invariant quantities under the evolution.*
- *Use this to rewrite the initial value problem as a single equation for* $x(t) = n_A(t)$ *with parameters* $\alpha = n_A(0) - n_B(0)$ *and* $\beta = n_{AB}(0) + n_A(0)$.
- *Use phase line analysis to determine the long time behavior of* $x(t)$.

Existence and Uniqueness

Failure of uniqueness

Classic example of non-uniqueness

$$\begin{cases} \dot{x} = x^{1/3} & \text{for } t > 0, \\ x(0) = 0. \end{cases} \tag{6}$$

Can solve by separation of variables to find three solutions already

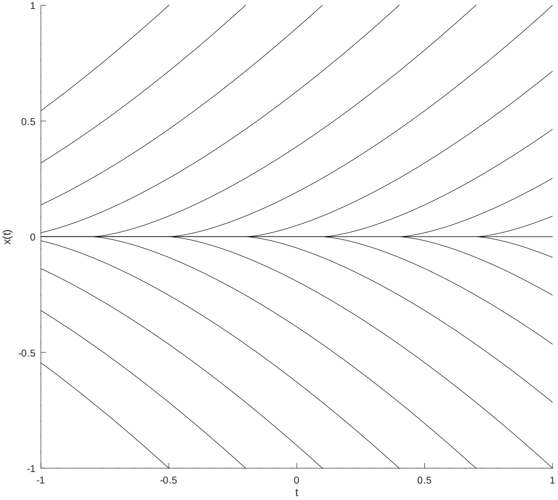
$$x(t) = \pm(2/3)^{3/2} t^{3/2} \;\text{ and }\; x(t) \equiv 0 \;\text{ for }\; t > 0.$$

However there are many more solutions

$$x(t) = \pm(2/3)^{3/2} \max\{0, (t - t_0)\}^{3/2}$$

are also solutions for any $t_0 \geq 0$.

Plot of non-uniqueness example



Non-uniqueness example

Remark

Note that the x derivative of $f(x) = x^{1/3}$ is not bounded near 0. As we will see soon this is exactly the cause of the non-uniqueness.

Remark

Note that this equation actually does actually have uniqueness *backwards* in time. However trajectories can cross in negative time which is equivalent to the forward in time non-uniqueness.

This is not "non-physical" see law for emptying a fluid filled cylinder by draining through a hole in the bottom $\dot{h} = -ah^{1/2}$ (after non-dimensionalization) where h is the height of fluid in the cylinder.

Failure of (global in time) existence

Classic example of failure of global existence

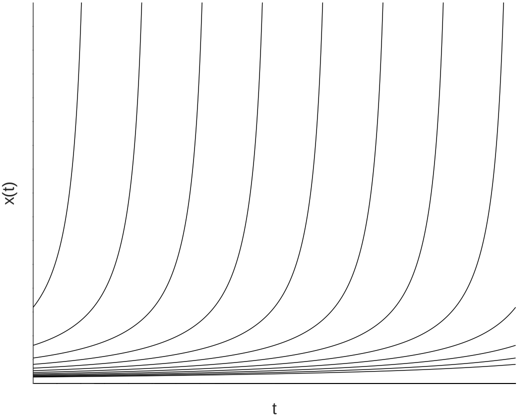
$$\begin{cases} \dot{x} = x^2 & \text{for } t > 0, \\ x(0) = x_0 > 0. \end{cases} \tag{7}$$

Typical separation of variables yields the solution

$$x(t) = \frac{x_0}{1 - x_0 t} \quad \text{for} \quad 0 \leq t < 1/x_0.$$

We say this solution **blows up** at time $1/x_0$.

Blow-up example



Integral form of IVP

Suppose we have a solution of the IVP

$$\begin{cases} \dot{x}(t) = f(t, x(t)) & \text{for } t \in (0, T) \\ x(0) = x_0 \end{cases} \tag{IVP}$$

(i.e. it is a C^1 function and satisfies the equation pointwise). By fundamental theorem of calculus

$$\begin{aligned} x(t) &= x_0 + \int_0^t \dot{x}(s)\, ds \\ &= x_0 + \int_0^t f(s, x(s))\, ds \end{aligned}$$

Integral form of IVP

Say that $x \in C([0, T] \to \mathbb{R})$, is a solution of the integral form of (IVP) if

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds \quad \text{for all} \quad t \in [0, T]. \qquad \text{(I-IVP)}$$

Now if x solves (I-IVP) then, since $f(s, x(s))$ is a continuous function, fundamental theorem of calculus implies x is differentiable and

$$\dot{x}(t) = f(t, x(t)) \quad \text{for all} \quad t \in [0, T].$$

Thus the integral and differential forms of the initial value problem are equivalent.

Integral form of IVP

Remark

This is a simple example of the general idea of **weak solution** which appears often in PDE. One introduces a weaker notion of solution which makes existence easier. Then one has to do some work to show uniqueness of weak solutions, or even better that weak solutions are **classical solutions** (easy in this case).

A useful note

Remark

If $x(t)$ is continuous on an interval $[a,b]$ and solves $\dot{x} = f(t, x(t))$ except possibly at some time $t_0 \in (a,b)$. Then actually x solves on the whole interval. The proof is just to use the integral forms

$$x(t) = x(a) + \int_0^t f(s, x(s))\, ds \quad \text{for} \quad a \leq t < t_0$$

and

$$x(t) = x(t_0) + \int_{t_0}^t f(s, x(s))\, ds \quad \text{for} \quad t_0 \leq t < b.$$

Solution concatenation

Lemma

If x is a solution of $\dot{x} = f(t,x)$ on $(a,0]$ and y is a solution of $\dot{y} = f(t,y)$ on $[0,b)$ and $x(0) = y(0)$ then

$$z(t) := \begin{cases} x(t) & a < t \leq 0 \\ y(t) & 0 \leq t \leq b \end{cases}$$

is a solution of $\dot{z} = f(t,z)$ on (a,b).

Uniqueness

Lipschitz continuity

A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is called **Lipschitz continuous** with **Lipschitz constant** L if

$$|g(x) - g(y)| \leq L|x - y| \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is called **locally Lipschitz continuous** if it is Lipschitz continuous on every compact subset of $\mathbb{R}^n$.

The Lipschitz constant

It turns out that g is Lipschitz continuous on a convex set $U \subset \mathbb{R}^n$ if and only if Dg is bounded on U (in measure theoretic sense).

A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is called differentiable at a point x_0 if there is a linear mapping $Dg(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ (i.e. an $n \times n$ matrix) so that

$$\lim_{x \to x_0} \frac{|g(x) - g(x_0) - Dg(x_0)(x - x_0)|}{|x - x_0|} = 0$$

(Note: I am writing $|v| = (v_1^2 + \cdots + v_n^2)^{1/2}$ for vectors $v \in \mathbb{R}^n$)

If g is continuously differentiable then

$$[Dg(x_0)]_{ij} = \frac{\partial g_i}{\partial x_j}(x_0).$$

The Lipschitz constant

For x and y in U the line $[x, y]$ from x to y is contained in U so

$$\begin{aligned}
|g(y) - g(x)| &= \left| \int_0^1 \frac{d}{ds}[g(x + s(y-x))] \; dt \right| \\
&= \left| \int_0^1 Dg(x + s(y-x))(y-x) \; ds \right| \\
&\leq \int_0^1 |Dg(x + s(y-x))(y-x)| \; ds \\
&\leq \int_0^1 \|Dg(x + s(y-x))\|_{op} |y-x| \; ds \\
&\leq \sup_{z \in [x,y]} \|Dg(z)\|_{op} |x-y|.
\end{aligned}$$

In particular when g is C^1 it is locally Lipschitz, if it is C^1 with bounded derivative then it is (globally) Lipschitz.

Uniqueness theorem

Theorem

Suppose $f(t,x)$ is Lipschitz continuous in x with constant L for all $t \geq 0$. Then if x and y are solutions of

$$\dot{x} = f(t,x) \;\; \text{and} \;\; \dot{y} = f(t,y) \;\; \text{for} \;\; t > 0$$

then

$$|x(t) - y(t)| \leq e^{Lt}|x_0 - y_0| \;\; \text{for} \;\; t \geq 0.$$

In particular if $x_0 = y_0$ then $x(t) = y(t)$.

Grönwall's Inequality

Lemma (Grönwall)

Suppose $A \geq 0$ *and* B, *and* ϕ *are continuous non-negative functions on* $[0, T]$ *and* ϕ *solves the integral inequality*

$$\phi(t) \leq A + \int_0^t B(s)\phi(s)\, ds \quad \textit{for all} \quad t \in [0, T] \tag{8}$$

then

$$\phi(t) \leq A \exp\left(\int_0^t B(s)\, ds\right).$$

Grönwall's Inequality Proof

Assume for now that $A > 0$. Define

$$u(t) := A + \int_0^t B(s)\phi(s)\ ds.$$

By fundamental theorem of calculus this function is C^1. Compute

$$\dot{u}(t) = B(t)\phi(t) \leq B(t)u(t).$$

Since $u > 0$ we can divide both sides by $u(t)$ and write

$$\frac{d}{dt}(\log u(t)) \leq B(t).$$

Grönwall's Inequality Proof

Integrating on both sides from 0 to t

$$\log u(t) \leq \log A + \int_0^t B(s)\, ds.$$

Then exponentiate to find

$$\phi(t) \leq u(t) \leq A \exp\Big(\int_0^t B(s)\, ds\Big).$$

For the case $A = 0$ note that ϕ also satisfies the integral inequality (8) for all $A > 0$, so we can just send $A \to 0$ in the above inequality. $\square$

Grönwall's Inequality Proof (slight variation)

Assume for now that $A > 0$. Define

$$u(t) := A + \int_0^t B(s)\phi(s)\ ds.$$

By fundamental theorem of calculus this function is C^1. Compute

$$\dot{u}(t) = B(t)\phi(t) \leq B(t)u(t).$$

Multiply both sides by $e^{-\int_0^t B(s)\ ds}$

$$e^{-\int_0^t B(s)\ ds}\dot{u}(t) - e^{-\int_0^t B(s)\ ds}B(t)u(t) \leq 0.$$

Grönwall's Inequality Proof (slight variation)

Now the left hand side is a derivative

$$\frac{d}{dt}(e^{-\int_0^t B(s)\ ds}u(t)) \leq 0.$$

Integrate from 0 to t to find

$$u(t) \leq u(0)\exp(\int_0^t B(s)\ ds)$$

and then conclude by using $\phi(t) \leq u(t)$ and $u(0) = A$. □

Uniqueness theorem (again)

Theorem

Suppose $f(t,x)$ is Lipschitz continuous in x with constant L for all $t \geq 0$. Then if x and y are solutions of

$$\dot{x} = f(t,x) \quad \text{and} \quad \dot{y} = f(t,y) \quad \text{for} \quad t > 0$$

then

$$|x(t) - y(t)| \leq e^{Lt}|x_0 - y_0| \quad \text{for} \quad t \geq 0.$$

In particular if $x_0 = y_0$ then $x(t) = y(t)$.

Uniqueness proof

Proof.

Using the integral formula

$$x(t) - y(t) = x_0 - y_0 + \int_0^t [f(s, x(s)) - f(s, y(s))] \, ds.$$

By triangle inequality

$$\begin{aligned} |x(t) - y(t)| &\leq |x_0 - y_0| + \int_0^t |f(s, x(s)) - f(s, y(s))| \, ds \\ &\leq |x_0 - y_0| + \int_0^t L|x(s) - y(s)| \, ds \end{aligned}$$

Apply Grönwall's inequality with $\phi(t) = |x(t) - y(t)|$. $\square$

Distinct solutions cannot cross

Corollary

Suppose $f(t,x)$ is Lipschitz continuous in x with constant L for all $t \geq 0$. If x and y are solutions of

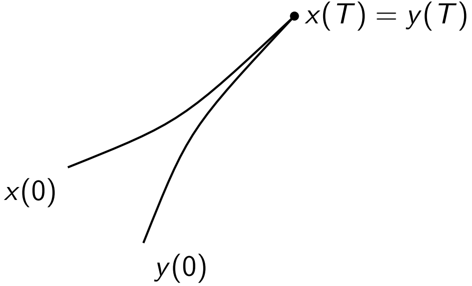
$$\dot{x} = f(t,x) \quad \textit{and} \quad \dot{y} = f(t,y) \quad \textit{for} \quad t > 0$$

then

$$|x(t) - y(t)| \geq e^{-Lt}|x_0 - y_0|.$$

Distinct solutions cannot cross

Idea: Run the equation backwards in time and apply the uniqueness result.



Distinct solutions cannot cross

Proof.

Idea: Run the equation backwards in time and apply the uniqueness result.

Fix $T > 0$ and consider the trajectories traced in the reverse direction $\bar{x}(t) = x(T - t)$ and $\bar{y}(t) = y(T - t)$. These solve the ODE

$$\dot{z} = -f(T - t, z)$$

with initial data $\bar{x}(0) = x(T)$ and $\bar{y}(0) = y(T)$. By the uniqueness theorem applied to this backwards ODE

$$\begin{aligned} |x(0) - y(0)| &= |\bar{x}(T) - \bar{y}(T)| \\ &\leq e^{LT}|\bar{x}(0) - \bar{y}(0)| \\ &\leq e^{LT}|x(T) - y(T)|. \end{aligned}$$

□

Autonomous equations

Definition

An ODE $\dot{x} = f(t,x)$ is called **autonomous** if the right hand side $f(t,x) = f(x)$ does not depend on t.

Remark

Note that any n-dimensional non-autonomous ODE system can be viewed as an $(n+1)$-dimensional autonomous system by solving for (t,x) in

$$\dot{t} = 1 \ \text{ and } \ \dot{x} = f(t,x).$$

This is not always a useful way to think.

Distinct trajectories of autonomous equations cannot cross

Definition

A **trajectory** of an autonomous ODE is the image of a solution as a subset of $\mathbb{R}^n$. That is, if $\dot{x} = f(x)$ on a time interval I then

$$\Gamma_x = x(I) = \{x(t) : \ t \in I\} \subset \mathbb{R}^n$$

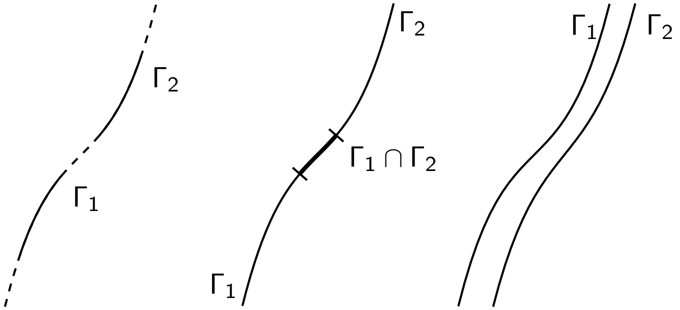
is a trajectory of the ODE. Trajectories of non-autonomous ODE are subsets of $(-\infty, \infty) \times \mathbb{R}^n$.

Lemma

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ globally Lipschitz continuous. If Γ_1 and Γ_2 are trajectories of $\dot{x} = f(x)$ then either they are disjoint or $\Gamma_1 \cup \Gamma_2$ is also a trajectory.

Understanding the no crossing lemma statement

Possible situations



No crossing proof

Suppose there is a point $z_0 \in \Gamma_x \cap \Gamma_y$, i.e. there is $t_x \in I_x$ and $t_y \in I_y$ such that $x(t_x) = z_0$ and $y(t_y) = z_0$. Define

$$\bar{x}(t) = x(t - t_x) \quad \text{and} \quad \bar{y}(t) = y(t - t_y),$$

by time translation invariance these also solve the same ODE and $\bar{x}(0) = \bar{y}(0) = z_0$. Thus by uniqueness $\bar{x}(t) = \bar{y}(t)$ on their common interval of definition.

Then, recalling our previous remark about continuous paths which solve the ODE except at finitely many times,

$$z(t) = \begin{cases} \bar{x}(t) & t \in I_x \\ \bar{y}(t) & t \in I_y \end{cases}$$

solves the ODE $\dot{z} = f(z)$ and $\Gamma_z = \Gamma_x \cup \Gamma_y$ is a trajectory. $\square$

Phase line analysis

First order autonomous equation $x(t) \in \mathbb{R}$

$$\begin{cases} \dot{x}(t) = f(x(t)) & t \in I \\ x(0) = x_0. \end{cases} \tag{9}$$

Lemma

If f is Lipschitz continuous, $f > 0$ on $[a, b]$, and $f(a) = f(b) = 0$ then for any $x_0 \in (a, b)$

$$\lim_{t \to -\infty} x(t) = a \quad \text{and} \quad \lim_{t \to \infty} x(t) = b.$$

Phase line analysis

Proof

Since a and b are stationary solutions we can apply the no crossing theorem and derive that $x(t) \neq a$ for all t and $x(t) \neq b$ for all t.

Since x is continuous and $x(0) \in (a, b)$ we must have $x(t) \in (a, b)$ for all $t \in \mathbb{R}$. This implies then that $\dot{x}(t) = f(x(t)) > 0$ so x is strictly monotone increasing. Thus the limits $\lim_{t \to \pm\infty} x(t)$ both exist.

Phase line analysis

Proof

If $x(t) \leq c < b$ for all $t > 0$ then

$$\inf_{t>0} f(x(t)) \geq \inf_{[x_0,c]} f \geq \mu > 0$$

since continuous functions achieve their minimum on compact sets and $f > 0$ on $[x_0, c]$. Then

$$b \geq x(t) = \int_0^t f(x(t))\, dt \geq \mu t$$

which is a contradiction for t large. Thus $\lim_{t\to\infty} x(t) = b$. $\square$

Some functional analysis (for Existence)

Normed spaces

Definition

Given a vector space V over $\mathbb{R}$ or $\mathbb{C}$ we say that $\|\cdot\| : V \to \mathbb{R}$ is a **norm** on V and call $(V, \|\cdot\|)$ a **normed vector space** if

- (Positivity) For all $v \in V$, $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.
- (Scaling) For all $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) and $v \in V$
$$\|\alpha v\| = |\alpha|\|v\|$$
- (Triangle inequality) For all $v, w \in V$
$$\|v + w\| \leq \|v\| + \|w\|.$$

Normed spaces

Definition

Examples of normed spaces

- Euclidean space $\mathbb{R}^n$ with the Euclidean norm

$$\|x\| = (\sum_{j=1}^{n} x_j^2)^{1/2}.$$

- The space $L(V)$ of linear operators $T : V \to V$ for a normed space $(V, \|\cdot\|_V)$ comes with a canonical norm called the **operator norm**

$$\|T\|_{L(V)} = \|T\|_{op} = \sup_{v \neq 0} \frac{\|T(v)\|_V}{\|v\|_V}.$$

Normed spaces

More examples of normed spaces

- The space of continuous functions $C(U)$ on a domain $U \subset \mathbb{R}^n$ with the supremum norm

$$\|f\|_{sup} = \sup_{x \in U} |f(x)|.$$

Definition

A normed space is called **complete** if every Cauchy sequence in V converges. A complete normed vector space is called a **Banach space**.

All of the previous spaces are complete (for the space of linear operators need to add the assumption that $(V, \|\cdot\|_V)$ is complete).

Contraction mappings

Definition

For a subset X of a normed space V a function $\phi : X \to X$ is called a **strict contraction** of X if there is $\mu < 1$ so that

$$\|\phi(x) - \phi(y)\| \leq \mu\|x - y\| \ \text{ for all } \ x, y \in X.$$

Theorem (Contraction mapping theorem)

If X is a closed subset of a complete normed space V and $\phi : X \to X$ is a strict contraction then ϕ has a fixed point, i.e. there is $x_* \in X$ such that

$$\phi(x_*) = x_*.$$

Contraction mapping theorem

Remark

Fixed point theorems in general and the contraction mapping theorem in particular are an extremely common and useful way to prove existence of solutions of equations of various types.

In particular we will apply it to achieve an existence theorem for ODE IVPs. As we will see the existence result will not be too abstract, actually the proof furnishes an algorithm with a convergence rate.

Proof of contraction mapping theorem

Let $x_0 \in X$ arbitrary then define iteratively

$$x_{n+1} = \phi(x_n).$$

Using the contraction property

$$\|x_{n+1} - x_n\| = \|\phi(x_n) - \phi(x_{n-1})\| \leq \mu \|x_n - x_{n-1}\|$$

so by induction

$$\|x_{n+1} - x_n\| \leq \mu^n \|\phi(x_0) - x_0\|.$$

Proof of contraction mapping theorem (ctd...)

Now for arbitrary $n > m$

$$\begin{aligned}\|x_n - x_m\| &\leq \sum_{j=m}^{n-1} \|x_{j+1} - x_j\| \\ &\leq \sum_{j=m}^{n-1} \mu^j \|\phi(x_0) - x_0\| \\ &\leq \frac{\mu^m}{1-\mu} \|\phi(x_0) - x_0\|.\end{aligned}$$

Thus x_n is a Cauchy sequence, and since X is closed and V is complete x_n converges to some $x_* \in X$. Then by continuity

$$x_* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \phi(x_n) = \phi(x_*).$$

□

Picard-Lindelöf Theorem (global version)

Theorem

Suppose $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfies*

$$|f(t,x) - f(t,y)| \leq L|x-y| \quad \textit{for all} \quad x,y \in \mathbb{R}^n.$$

Then for any $x_0 \in \mathbb{R}^n$ *there is a unique* C^1 *solution* $x(t)$ *of*

$$x(t) = x_0 + \int_0^t f(s,x(s))\, ds \quad \textit{for all} \quad t \in \mathbb{R}.$$

Outline of the proof

- Local existence
  - Define a functional whose fixed points are solutions of the ODE IVP
  - Show that this functional is a contraction if the time interval $[0, T]$ is sufficiently small depending on L.
- Global existence
  - ODE solutions can be extended by concatenation.
  - Apply local existence result repeatedly.

Proof of Picard-Lindelöf Theorem

First we define a space where we will search for solutions

$$X = \{x \in C([0,T] \to \mathbb{R}^n) : \ x(0) = x_0\}.$$

This is a closed subset of the Banach space $C([0,T] \to \mathbb{R}^n)$.

For each $x \in X$ define $\Phi : X \to X$ by

$$\Phi[x](t) := x_0 + \int_0^t f(s, x(s))\, ds \quad \text{for} \quad t \in [0,T].$$

Note that any fixed point of Φ is a solution of (I-IVP). We claim that for $T \leq \frac{1}{2L}$ the mapping Φ is a contraction of X.

Aside: example of the Picard iteration

For example if the equation is the scalar linear problem $\dot{x} = ax$ then the contraction mapping iteration on Φ produces

$$y_0(t) = x_0, \; y_1(t) = x_0 + \int_0^t a y_0(s) \, ds = x_0 + t a x_0$$

and so on

$$y_k(t) = x_0 + \int_0^t a y_{k-1}(s) \, ds = x_0 + x_0 a t + \cdots + x_0 \frac{a^k t^k}{k!}.$$

Of course this is the series expansion for the exponential so

$$y_k(t) \to y(t) = x_0 e^{at} \quad \text{as} \quad k \to \infty.$$

(Note: this argument also works exactly the same for linear systems, although we need to understand the matrix exponential)

Proof of Picard-Lindelöf Theorem (ctd...)

$\Phi(X) \subset X$

First we need to check that Φ indeed maps X to X, note that

$$\Phi[x](0) = x_0 + \int_0^0 f(s, x(s)) \, ds = x_0$$

so the initial data is correct. Also $\Phi[x](t)$ is continuous in t because it is the anti-derivative of a continuous function and hence is C^1.

Proof of Picard-Lindelöf Theorem (ctd...)

Contraction property

Now consider two paths $x, y \in X$ and we estimate for $0 \leq t \leq T$

$$\begin{aligned} |\Phi[x](t) - \Phi[y](t)| &= \Big| \int_0^t f(s, x(s)) - f(s, y(s)) \; ds \Big| \\ &\leq \int_0^t L|x(s) - y(s)| \; ds \\ &\leq LT\|x - y\|_{sup}. \end{aligned}$$

Since this inequality holds for all $t \in [0, T]$ we also find, using $T \leq \frac{1}{2L}$

$$\|\Phi[x] - \Phi[y]\|_{sup} \leq LT\|x - y\|_{sup} \leq \frac{1}{2}\|x - y\|_{sup}.$$

Thus Φ is a contraction and has a (unique) fixed point in X.

Proof of Picard-Lindelöf Theorem (ctd...)

Global extension

Now suppose that we have a solution x of (IVP) on some interval $[0, T]$. By the previous argument there exists a (unique) solution of

$$\dot{y}(t) = f(t, y(t)) \text{ and } y(T) = x(T) \text{ for } t \in [T, T + \frac{1}{2L}].$$

By the solution concatenation lemma

$$z(t) = \begin{cases} x(t) & t \in [0, T] \\ y(t) & t \in [T, T + \frac{1}{2L}] \end{cases}$$

is also a solution of (IVP) now on $[0, T + \frac{1}{2L}]$. By iteration there is a solution on the time interval $[0, \infty)$. □

Localizing existence and uniqueness results

Locally Lipschitz

It is not very convenient that our only uniqueness result requires a *global* assumption (Lipschitz continuity) to get a *local* result (uniqueness).

Definition

Call $f(t,x)$ to be **locally Lipschitz continuous uniformly in** t in a domain $U \subset \mathbb{R}^n$ if for every $K \subset U$ compact and every $T > 0$ there is an $L \geq 1$ so that

$$|f(t,x) - f(t,y)| \leq L|x-y| \quad \text{for} \quad x, y \in K \quad \text{and} \quad t \in [-T, T].$$

Example

- $f(x) = x^2$
- Any $f(t,x)$ which is in the space $C([0,T]; C^1(U))$.

Localizing the uniqueness theorem

Theorem

Suppose $f(t,x)$ is locally Lipschitz continuous uniformly in t on $(t,x) \in \mathbb{R} \times U$ then any two solutions of the initial value problem

$$\dot{x} = f(t,x) \ \ \textit{and} \ \ x(0) = x_0$$

agree on their common time interval of definition.

Uniqueness proof

Uniqueness is a "local property" let's first show how to reduce to the case of proving uniqueness on a short time interval.

Suppose that x and y are two solutions of the IVP defined, respectively, on open time intervals I_x and I_y. Call

$$E = \{t \in I_x \cap I_y : \ x(t) = y(t)\}.$$

If we show that E is both open and closed then it must be the whole interval.

Uniqueness proof

The set E is immediately closed because it is the set where the difference of two continuous functions is zero.

To show E is open we must show that if some time $t_0 \in E$, i.e. $x(t_0) = y(t_0)$, then $x(t) = y(t)$ in an open neighborhood $(t_0 - \delta, t_0 + \delta)$. *This is local version of the current uniqueness theorem, we can just argue with $t_0 = 0$.*

Uniqueness proof

Now given $x_0 \in U$ choose $r > 0$ so that $\overline{B_r(x_0)} \subset U$. Then there is $L \geq 1$ so that $f(t,x)$ is Lipschitz continuous in x with constant L on $\overline{B_r(x_0)}$ for every $-1 \leq t \leq 1$.

Since f is a continuous function on the compact set $[-1,1] \times \overline{B_r(x_0)}$ there is an $M \geq 1$ so that $|f| \leq M$ on that set.

Call $t_* = \sup\{t : x([0,t]) \subset \overline{B_r(x_0)}\}$, in words t_* *is the first time that* $x(t)$ *leaves* $\overline{B_r(x_0)}$. We want to show that $t_* \geq \delta := r/M$. If $t_* = +\infty$ we are done, otherwise $x(t_*) \in \partial B_r(x_0)$ and so

$$r = |x(t_*) - x_0| \leq Mt_*$$

in which case $t_* \geq M/r = \delta$.

Thus any solution of the ODE IVP starting from x_0 stays in $\overline{B_r(x_0)}$ for $0 \leq t \leq \delta$ and we can apply the original uniqueness proof with the Lipschitz constant L.

Maximal interval of existence

Theorem

Suppose $U \subset \mathbb{R}^n$ and f is locally Lipschitz continuous on U uniformly in t. Then for any $x_0 \in U$ there is an open interval $I(x_0) \subset \mathbb{R}$ containing $t = 0$ and a unique C^1 solution $x(t)$ of

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds \quad \text{and } x(t) \in U \text{ for all } \; t \in I(x_0)$$

so that $I(x_0) = (t_-, t_+)$ is maximal in the following sense: for any compact $K \subset U$

$$x(t) \notin K \quad \text{for} \quad \min_{\pm} |t - t_\pm| \quad \text{sufficiently small.}$$

Examples

We have already seen a nice example

$$\dot{x} = x^2 \quad \text{and} \quad x(0) = x_0 > 0.$$

Then the solution is

$$x(t) = \frac{x_0}{1 - x_0 t} \quad \text{for} \quad t < \frac{1}{x_0}.$$

In this case the maximal interval of existence is easy to read off $I(x_0) = (-\infty, \frac{1}{x_0})$.

Examples

Consider the system in $U = \{(x, y, z) : z > 0\} \subset \mathbb{R}^3$

$$\dot{x} = -\frac{y}{z^2}, \;\; \dot{y} = \frac{x}{z^2}, \quad \text{and} \quad \dot{z} = 1 \tag{10}$$

with initial data

$$(x(0), y(0), z(0)) = (0, -1, \frac{1}{\pi}).$$

Can check explicitly the solution is

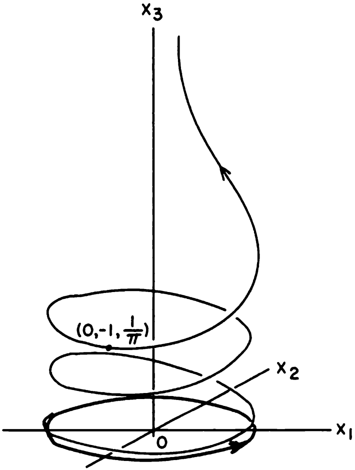
$$(\sin\frac{1}{t}, \cos\frac{1}{t}, t) \quad \text{on} \quad t \in (0, \infty)$$

which is the maximal interval of existence.

Perko, Chapter 2.4, Example 3

Examples

$$(\sin\frac{1}{t}, \cos\frac{1}{t}, t) \quad \text{on} \quad t \in (0,\infty)$$



Perko, Chapter 2.4, Example 3

Non-trivial continuation interval on each compact subset

Lemma

Suppose $U \subset \mathbb{R}^n$ open and $f : \mathbb{R} \times U \to \mathbb{R}^n$ satisfies for any compact $K \subset U$ there is L so that

$$|f(t,x) - f(t,y)| \leq L|x - y| \quad \text{for all } \ x, y \in K \quad \text{and all } t.$$

Suppose that $K \subset U$ compact. There exists $T(K) > 0$ so that for all $x_0 \in K$ there exists a unique solution of (IVP) for

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds \ \text{ and } x(t) \in U \text{ for all } \ t \in (-T(K), T(K)).$$

Sketch.

There is an $r > 0$ so that for all $x_0 \in K$ have $\overline{B_r(x_0)} \subset U$, call $K^r = \cup_{x_0 \in K} \overline{B_r(x_0)}$ which is also a compact subset of U. Then localize the domain for the fixed point argument:

$$X = \{x : [0, T] \to \overline{B_r(x_0)} : \quad x(t) \text{ continuous and } x(0) = x_0\}.$$

Define L to be the Lipschitz constant of $f(t, \cdot)$ on K^r and

$$M = \sup_{t \in [0,1], x \in K^r} |f(t, x)|.$$

For T sufficiently small depending on L and M the Picard fixed point functional will map X to itself (a non-trivial issue now) and be a contraction. □

Maximal interval of existence

Define

$$t_+ = \sup\{t > 0 : \exists \text{ a solution of (I-IVP) on } [0, t]\}$$

and

$$t_- = \inf\{t < 0 : \exists \text{ a solution of (I-IVP) on } [t, 0]\}.$$

There is a solution $x(t)$ of (I-IVP) on $I = (t_-, t_+)$ (exercise).

Maximal interval of existence

Suppose that there is a compact set $K \subset U$ and a sequence of times $(t_n)_{n\in\mathbb{N}}$ with $t_n \to t_+$ so that

$$x(t_n) \in K.$$

By the non-trivial continuation Lemma there is a time $T(K) > 0$ so that any initial data in K has a solution for at least time $T(K)$. Let n sufficiently large so that $t_+ - t_n < T(K)$. Then define $y(t)$ to be the solution of

$$\dot{y}(t) = f(t, y(t)) \text{ and } y(t_n) = x(t_n)$$

which exists at least until $t_n + T(K) > t_+$. The concatenation of $x(t)$ and $y(t)$ is a solution of the IVP on $[0, t_n + T(K)]$ which contradicts the definition of t_+. $\square$

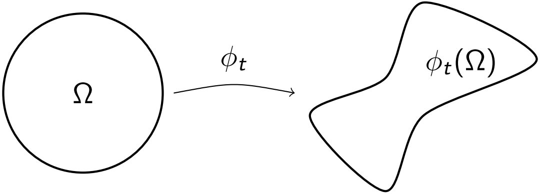
The flow map and invariants

Flow map

In the following let's restrict to the easy setting f globally Lipschitz continuous in x uniformly in t.

The existence and uniqueness theorems allow us to define the **flow map** $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$. For each $x_0 \in \mathbb{R}^n$ let $\phi_t(x_0)$ be the solution of the ODE IVP

$$\frac{d}{dt}\phi_t(x_0) = f(t, \phi_t(x_0)) \quad \text{with} \quad \phi_0(x_0) = x_0.$$



Flow map regularity

Quantitative uniqueness results imply regularity / invertibility of the flow map.

Theorem

If $f(t,x)$ is Lipschitz continuous in x with constant L for all t then

$$e^{-Lt}|x_0 - y_0| \leq |\phi_t(x_0) - \phi_t(y_0)| \leq e^{Lt}|x_0 - y_0|.$$

In particular $\phi_t(x)$ is Lipschitz in x with constant e^{Lt}.

Flow maps of autonomous systems

Flow maps of autonomous systems, such as

$$\frac{d}{dt}\phi_t(x_0) = f(\phi_t(x_0)) \quad \text{with} \quad \phi_0(x_0) = x_0,$$

have some very nice properties:

- (Initial identity)
$$\phi_0(x) = x.$$
- (Group property) For all t, s and x
$$\phi_{t+s}(x) = \phi_t(\phi_s(x)) = \phi_s(\phi_t(x)).$$
- (Inversion by backwards flow) For all t
$$\phi_{-t}(\phi_t(x)) = x.$$

Invariant quantities

A quantity $H : \mathbb{R}^n \to \mathbb{R}$ is called an **invariant quantity under the flow** ϕ_t if

$$H(\phi_t(x)) = H(x) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Or, in other words, if

$$0 = \frac{d}{dt} H(\phi_t(x)) = \nabla H(\phi_t(x)) \cdot \dot{\phi}_t(x) = \nabla H(\phi_t(x)) \cdot f(\phi_t(x)).$$

Example

- $H(p, x) = \frac{1}{2} p^2 + V(x)$ for the Newton's equations / Hamiltonian system

$$\dot{x} = p \quad \text{and} \quad \dot{p} = -\nabla V(x).$$

Invariant sets

A subset of phase space $S \subset \mathbb{R}^n$ is called an **invariant set under the flow** ϕ_t if

$$\phi_t(S) \subset S$$

for all $t \in \mathbb{R}$. It is called **positively invariant** if this holds for all $t > 0$, and **negatively invariant** if this holds for all $t < 0$.

Example

- Any level set $\{H(x) = c\}$ of an invariant quantity H, e.g. the total energy for Newton's equations.
- Any solution trajectory.

Invariant sets: 2×2 linear system (saddle)

The linear system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

The equations are uncoupled so we can solve

$$x(t) = x_0 e^{-t} \quad \text{and} \quad y(t) = y_0 e^{t}.$$

Then every level surface

$$xy = c$$

in $\mathbb{R}^2$ is an invariant set.

Invariant sets: 2×2 linear system (saddle)

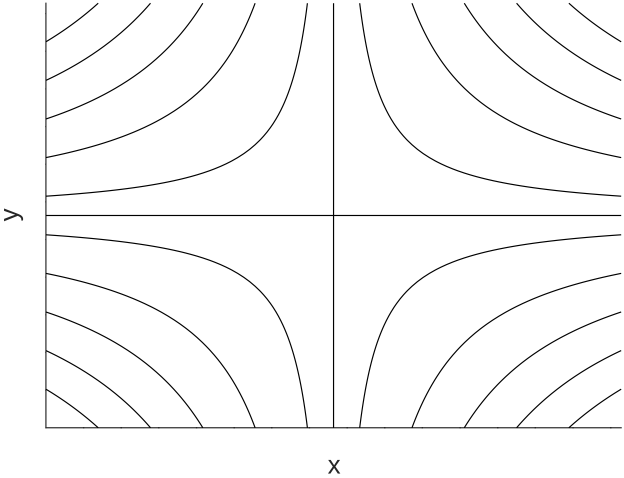


Figure: The axes are invariant, but note that no neighborhood of the origin is positively or negatively invariant.

Invariant sets: 2×2 non-linear saddle

The non-linear system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y + x^2 \end{bmatrix}.$$

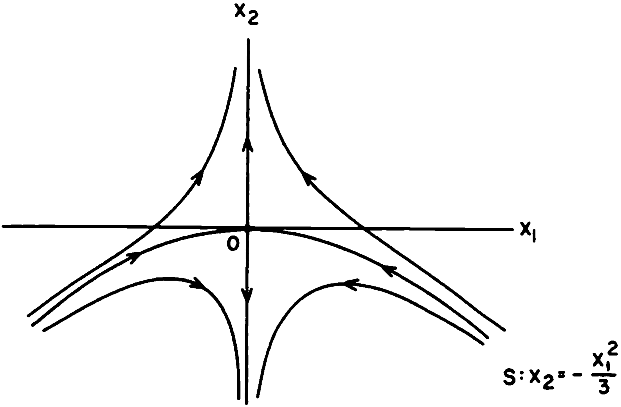
Easy to solve explicitly, solve first equation and then plug into second

$$\phi_t(x_0, y_0) = \begin{bmatrix} x_0 e^{-t} \\ y_0 e^t + \frac{x_0^2}{3}(e^t - e^{-2t}) \end{bmatrix}.$$

The $x_0 = 0$ axis is invariant, and also can check that the set $S = \{y_0 = -x_0^2/3\}$ is also invariant.

Perko, Chapter 2.5, Example 2

Invariant sets: 2×2 non-linear saddle



Perko, Chapter 2.5, Figure 4

Invariant sets: dissipative system

Positive/negative invariant sets naturally arise in dissipative systems. For example let's consider again the example of a mass-spring system with damping

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\frac{k}{m}x - \frac{\mu}{m}v \end{bmatrix}.$$

The energy

$$H(v,x) = \frac{1}{2}mv^2 + kx^2$$

is invariant when $\mu = 0$ and is decreasing in time when $\mu > 0$. In particular any sub-level set of the energy is positively invariant

$$S = \{(v,x) \in \mathbb{R}^2 : \; H(v,x) \leq \lambda\} \quad \text{for} \quad \lambda \geq 0.$$

Damped spring

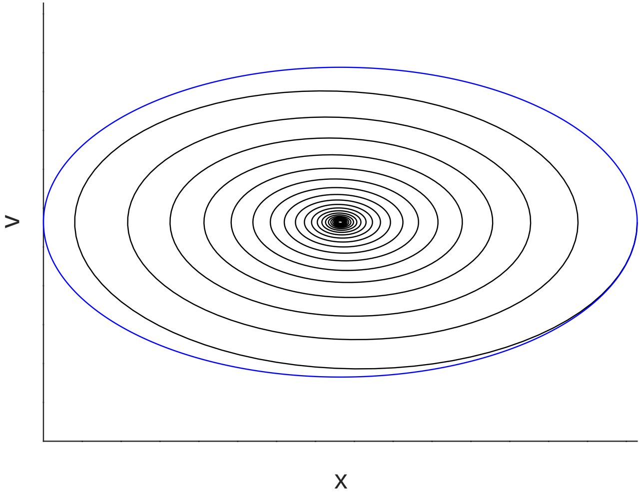


Figure: In blue the $\frac{1}{4}$ level set of the energy $H(v,x) = \frac{1}{2}v^2 + \frac{1}{4}x^2$. In black a solution of the damped mass-spring system started at $(x_0, v_0) = (1, 0)$.

Volume distortion by the flow map

One way to measure how trajectories of the ODE system are converging or diverging is by looking at how volumes change under the flow map. Recall the change of variables theorem tells us that for any Ω (measurable)

$$|\phi_t(\Omega)| = \int_\Omega |\det(D\phi_t(x))| \, dx$$

Using the inequality

$$\|A^{-1}\|_{op}^n \leq \det(A) \leq \|A\|_{op}^n$$

we can find

$$e^{-nLt}|\Omega| \leq |\phi_t(\Omega)| \leq e^{nLt}|\Omega|.$$

Evolution of the Jacobian determinant

To be more precise we need to compute the ODE solved by the **Jacobian determinant of the flow map**

$$J(t,x) = \det(D\phi_t(x)).$$

It turns out, following some computations, that the Jacobian determinant solves the following ODE IVP

$$\frac{d}{dt}J(t,x) = (\mathrm{div}_x f)(t,\phi_t(x))J(t,x) \quad \text{and} \quad J(0,x) = 1.$$

Evolution of the Jacobian determinant

$$\frac{d}{dt}J(t,x) = (\mathrm{div}_x f)(t,\phi_t(x))J(t,x) \ \text{ and } \ J(0,x) = 1.$$

Remark

If $\mathrm{div}_x(f(t,x)) \equiv 0$ then $J(t,x) \equiv 1$ and volumes do not change under the flow. If $\mathrm{div}(f(t,x)) \geq 0$ then $J(t,x) \geq 1$ for all $t > 0$ and volumes are increased under the flow, vice versa for negative divergence.

Remark

This formula is quite important in basic fluid mechanics. It explains why divergence free flows, as appear for example in the incompressible Euler or Navier Stokes equations, are actually *incompressible*.

Evolution of the derivative

Let's compute the ODE for $J(t, x)$. We start with the evolution of $D\phi_t(x)$:

$$\begin{aligned} \frac{d}{dt} D\phi_t(x) &= D(\frac{d}{dt}\phi_t(x)) \\ &= D(f(t, \phi_t(x))) \\ &= Df(t, \phi_t(x)) D\phi_t(x). \end{aligned}$$

Note that $D\phi_t(x)$ solves a *linear non-autonomous* ODE.

Derivative of the determinant

Now we need to compute the derivative of the determinant. Start with the derivative at the identity

$$\frac{d}{ds}\det(I + sB) = \lim_{s \to 0} \frac{\det(I + sB) - \det(I)}{s}.$$

The determinant $\det(I + sB)$ is a polynomial in s with zeroth order term 1 and linear term $\mathrm{tr}(B)s$:

$$\det(I + sB) = \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} (\delta_{i\sigma(i)} + sB_{i\sigma(i)})$$

Thus

$$\det{}'(I)(B) = \mathrm{tr}(B).$$

Derivative of the determinant

Now for the derivative of the determinant at an invertible matrix A

$$\begin{aligned}\det{}'(A)(B) = \frac{d}{ds}\det(A+sB) &= \frac{d}{ds}\det(A(I+sA^{-1}B))\\ &= \det(A)\frac{d}{ds}\det(I+sA^{-1}B))\\ &= \det(A)\mathrm{tr}(A^{-1}B).\end{aligned}$$

So if we are trying to compute the time derivative of an evolving determinant we can do so by chain rule, called **Jacobi's formula**

$$\frac{d}{dt}\det(A(t)) = \det{}'(A(t))(\frac{d}{dt}A(t)) = \det(A(t))\mathrm{tr}(A^{-1}\frac{d}{dt}A(t)).$$

Evolution of the Jacobian determinant

Combining the work on the previous slides

$$\begin{aligned}
\frac{d}{dt}J(t,x) &= J(t,x)\mathrm{tr}(D\phi_t(x)^{-1}\frac{d}{dt}D\phi_t(x)) \\
&= J(t,x)\mathrm{tr}(D\phi_t(x)^{-1}Df(t,\phi_t(x))D\phi_t(x)) \\
&= J(t,x)\mathrm{tr}(Df(t,\phi_t(x))D\phi_t(x)D\phi_t(x)^{-1}) \\
&= J(t,x)\mathrm{tr}(Df(t,\phi_t(x))) \\
&= J(t,x)(\mathrm{div}_x f)(t,\phi_t(x)).
\end{aligned}$$

We used the cyclic property of the trace on the third line.