

MATH 6410: Ordinary Differential Equations

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Initial value problems

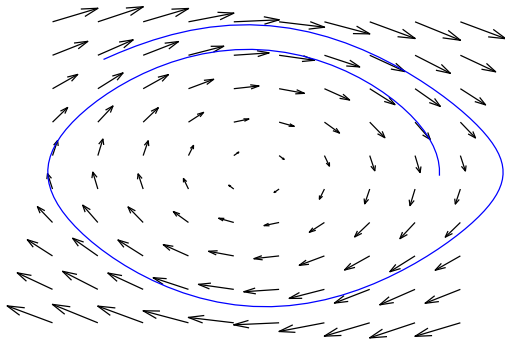
Initial value problems

Initial value problem (IVP) for a first order system: find $x(t) : [0, T] \rightarrow \mathbb{R}^n$ solving

$$\begin{cases} \dot{x}(t) = f(t, x(t)) & \text{for } t \in (0, T) \\ x(0) = x_0. \end{cases} \quad (1)$$

where $f : [0, T] \times U \rightarrow \mathbb{R}^n$ is continuous, $U \subset \mathbb{R}^n$ open, and $x_0 \in U$.

Following a vector field



The fundamental questions

Mathematical theory of differential equations always starts with a set of questions called **well-posedness**

- ▶ **Existence:** “Is there a solution?”
 - ▶ Local (i.e. small interval around initial time)
 - ▶ Global (all positive times)
- ▶ **Uniqueness:** “Is there only one solution?”
 - ▶ Regularity of the dependence on the data.
 - ▶ Regularity of the dependence on the equation.

Qualitative features

Once we have understood the basics we can begin to ask more refined questions and start to really understand the solutions of a particular ODE.

- ▶ Long time / asymptotic behavior of solutions
- ▶ Dependence on the equation
 - ▶ Bifurcation theory
 - ▶ Perturbation theory

Higher order equations

What about differential equations involving higher order derivatives? Why just consider first order systems?

General abstract higher order ODE, look for $y(t)$ solving

$$\begin{cases} F(y^{(k)}, \dots, y^{(1)}, y, t) = 0 & \text{for } t > 0 \\ y^{(j)}(0) = y_{0,j} & \text{for } 0 \leq j \leq k-1. \end{cases} \quad (2)$$

If $\partial_1 F(\vec{y}_0, 0) \neq 0$ then, by implicit function theorem, can solve for $y^{(k)}$ (locally)

$$y^{(k)}(t) = g(t, y(t), \dots, y^{(k-1)}(t))$$

Higher order equations (ctd...)

Now write

$$x(t) = (y(t), \dots, y^{(k-1)}(t))^T \in \mathbb{R}^k$$

Which solves the first order system

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(0) = (y(0), y^{(1)}(0), \dots, y^{(k-1)}(0))^T \end{cases} \quad \text{for } t > 0 \quad (3)$$

with

$$f(t, x) = (x_2, x_3, \dots, x_{k-1}, g(t, x_1, \dots, x_k))^T$$

Examples and ideas

Newton's Equations

Particle of mass $m > 0$ at position $x(t)$ moves in $\mathbb{R}$ under the influence of a force field $F : \mathbb{R} \rightarrow \mathbb{R}$ and a kinetic frictional force (coefficient $\mu \geq 0$) opposing motion

$$m\ddot{x} = -\mu\dot{x} + F(x).$$

It is common to view this as a first order system by considering the equation for $(x, p) = (x, m\dot{x})$ the position and momentum

$$\begin{cases} \dot{x} = \frac{1}{m}p \\ \dot{p} = -\frac{\mu}{m}p + F(x). \end{cases} \quad (4)$$

Newton's Equations

Newton's equations have a conserved ($\mu = 0$) / dissipated ($\mu > 0$) quantity. Multiply the ODE by $\dot{x}$

$$m\ddot{x}\dot{x} - \dot{x}F(x) = -\mu\dot{x}^2$$

and note that the left hand side is a derivative

$$\frac{d}{dt}\left(\frac{p^2}{2m} + V(x)\right) = -\mu\dot{x}^2 \leq 0$$

where

$$V(x) = -\int_0^x F(u) \, du.$$

Newton's Equations

This quantity is called the **Hamiltonian**

$$H(p, x) = \frac{p^2}{2m} + V(x)$$

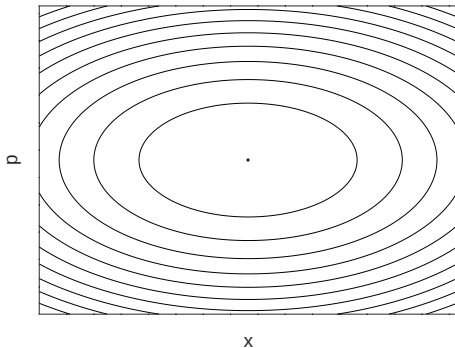
When $\mu = 0$ the solutions of Newton's equations remain on level sets of the Hamiltonian and the first order system has the **Hamiltonian form**

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p}(p, x) \\ \dot{p} = -\frac{\partial H}{\partial x}(p, x). \end{cases}$$

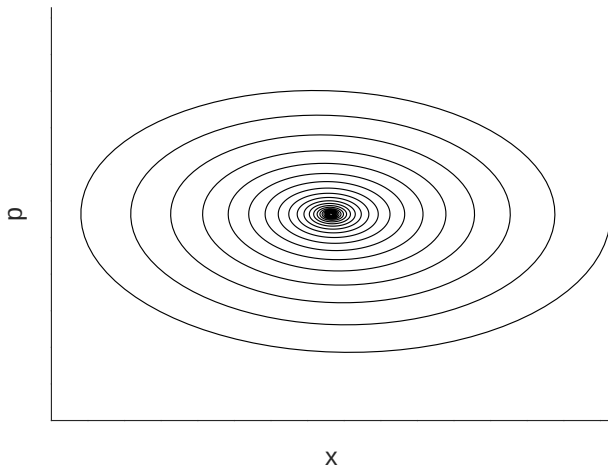
Newton's equations: mass on spring

Ideal point mass m on a spring with constant k , rest state at 0,

$$H(p, x) = \frac{p^2}{2m} + \frac{1}{2}kx^2.$$



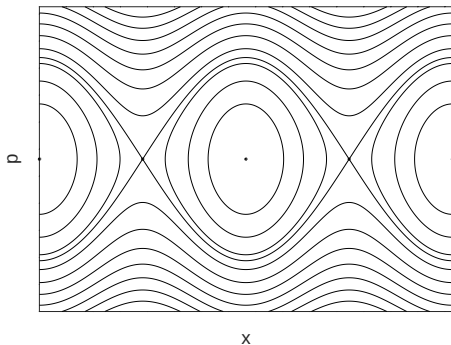
Damped mass on spring



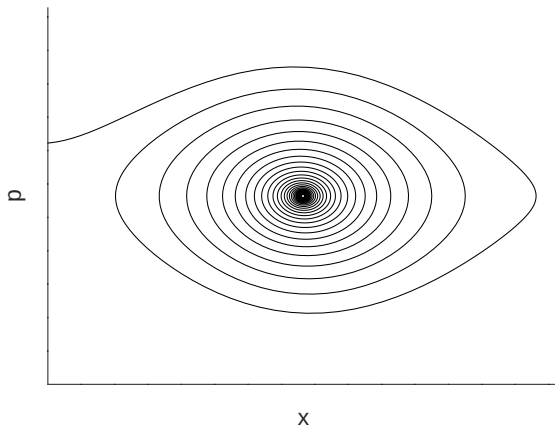
Newton's equations: pendulum

Ideal point mass m hung on a string of length L , θ is the angle from the vertical axis and $p = mL\dot{\theta}$ is the momentum

$$H(p, x) = \frac{p^2}{2m} + mgL(1 - \cos \theta).$$



Damped pendulum



Population dynamics

Most basic equation of population dynamics is the most basic ODE of all

$$\dot{N} = rN \quad \text{and} \quad N(0) = N_0.$$

Models, for example, population growth of some species, nuclear decay, etc. The number r is the reproduction / decay rate.

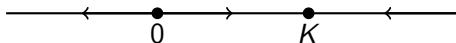
Slightly more sophisticated is the logistic growth model

$$\dot{N} = rN(1 - N/K) \quad \text{and} \quad N(0) = N_0.$$

Here K is the carrying capacity.

Phase line analysis

Plot the positive/negative regions of $f(N) = rN(1 - N/K)$ on $\mathbb{R}$



Solutions converge to **critical point** (zero of f and **stationary solution** of the ODE) either 0 or K based on where initial data sits. We will make this rigorous later.

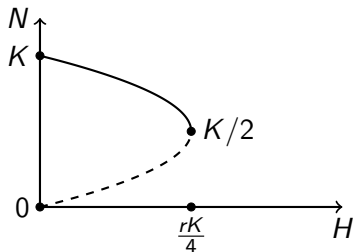
Population dynamics

Now if we add in “harvesting” to the logistic growth model

$$\dot{N} = rN(1 - N/K) - H \quad \text{and} \quad N(0) = N_0$$

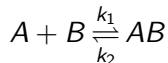
where H is the number of the population harvested per unit time.

For $H < \frac{rK}{4}$ there are two positive fixed points, which collide and annihilate at $H = \frac{rK}{4}$, called a **saddle-node bifurcation**.



Chemical reaction

Reversible synthesis/decomposition reaction between chemicals A and B producing AB



forward reaction occurs with rate k_1 backwards occurs with rate k_2 .

Call $n_X(t)$ to be the concentration of reactant X at time t , then the densities evolve by the ODE system

$$\begin{cases} \dot{n}_{AB} = k_1 n_A n_B - k_2 n_{AB} \\ \dot{n}_A = \dot{n}_B = k_2 n_{AB} - k_1 n_A n_B. \end{cases} \quad (5)$$

Chemical reactions: homework

Problem

Consider the chemical reaction system

$$\begin{cases} \dot{n}_{AB} = k_1 n_A n_B - k_2 n_{AB} \\ \dot{n}_A = \dot{n}_B = k_2 n_{AB} - k_1 n_A n_B. \end{cases}$$

- ▶ *Show that $n_A - n_B$, $n_{AB} + n_A$, $n_{AB} + n_B$, and $n_{AB} + \frac{1}{2}(n_A + n_B)$ are all invariant quantities under the evolution.*
- ▶ *Use this to rewrite the initial value problem as a single equation for $x(t) = n_A(t)$ with parameters $\alpha = n_A(0) - n_B(0)$ and $\beta = n_{AB}(0) + n_A(0)$.*
- ▶ *Use phase line analysis to determine the long time behavior of $x(t)$.*