HOMEWORK PROBLEMS MATH 6420

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Each problem shows the date assigned. Problems are due approximately one and a half weeks after the assignment date, each problem will have a separate gradescope assignment which you can use to help keep track of due dates. See the most up to date syllabus for more details about late submissions etc.

Problem 1 (Jan 14)**.** Solve the following initial value / initial boundary value problems using the method of characteristics. I usually find it helpful to draw a picture of the space-time domain and the characteristics.

(a) Let $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ find the general solution of,

$$\begin{cases} u_t + b \cdot D_x u = cu & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & x \in \mathbb{R}^n. \end{cases}$$

Hint: Take the derivative of $z(s) = u(x+sb, t+s)$ as we did in class, just $\dot{z}$ will not equal to zero now.

(b) Find the general solution of

$$\begin{cases} u_t + xu_x = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = g(x) & \text{in } \mathbb{R}. \end{cases}$$

Hint: Look for a path $x(t)$ so that $\frac{d}{dt}u(x(t),t) = 0$ (instead of the straight line paths from class).

(c) Let $b > 0$ find the general solution of,

$$\begin{cases} u_t + bu_x = 0 \ \text{ for } \ (x,t) \in (0,\infty) \times (0,\infty) \\ u(x,0) = f(x) \ \text{ and } \ u(0,t) = g(t). \end{cases}$$

How would your solution change if $b < 0$.

Hint: Draw the space-time domain, draw the characteristics (lines parallel to $(b,1)$) and think about where they cross the boundary of the space-time domain.

(d) Find the general solution of,

$$\begin{cases} u_t + x^{1/2} u_x = 0 \ \text{ for } \ (x,t) \in (0,\infty) \times (0,\infty) \\ u(x,0) = f(x) \ \text{ and } \ u(0,t) = g(t) \end{cases}$$

Problem 2 (Jan 14)**.** Use the method of characteristics to solve the following equation

$$\begin{cases} u_t + x|x|u_x = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R}. \end{cases}$$

After finding the general solution, suppose additionally that $f(x) = 0$ for $|x| \leq r$. Show that there is minimal time $T(r)$ so that for any such f the corresponding solution u satisfies $u(\cdot,t) \equiv 0$ for all $t \geq T$, calculate $T(r)$.

Problem 3 (Jan 19)**.** [Evans, 2nd edition, Ch. 2 Problem 5] Let U be a bounded domain of $\mathbb{R}^n$. We say $u \in C^2(U)$ is *subharmonic* if

$$-\Delta u \leq 0 \ \text{ in } \ U.$$

(a) Prove for subharmonic v that
$$v(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} v(y)\ dy \ \text{ for all } \ \overline{B(x,r)} \subset U.$$
(b) Prove that the weak maximum principle holds for subharmonic $v \in C^2(U) \cap C(\overline{U})$.
(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v = \phi(u)$. Prove that v is subharmonic.
(d) Prove $v = |Du|^2$ is subharmonic whenever **[EDIT]** u **is harmonic**(you can assume that $|Du|^2$ is $C^2(U)$).

Problem 4 (Jan 19)**.** We say that $u \in C(U)$ is *weakly harmonic* in U if for every $\phi \in C_c^\infty(U)$

$$\int_U u(x)\Delta\phi(x)\ dx = 0.$$

Show that if u is weakly harmonic then u is C^2 and harmonic.

Hint: Read Appendix C.5 in Evans on the topic of "Convolution and smoothing".

Problem 5 (Jan 21)**.** Let U be a bounded domain of $\mathbb{R}^n$ and $u \in C^2(U) \cap C(\overline{U})$ which is harmonic in U. Suppose that $u(x_0) = \min_{\overline{U}} u = 0$ at some $x_0 \in \partial U$. Suppose that U has an interior tangent ball at x_0, that is there exists x_1 so that $B(x_1,r) \subset U$ and $\partial B(x_1,r) \cap \partial U = \{x_0\}$. Prove that if u is not constant then,

$$\frac{\partial u}{\partial \nu}(x_0) < 0,$$

where ν is the outward unit normal to $B(x_1,r)$ at x_0.

Note: I have not assumed enough to guarantee that the normal derivative $\frac{\partial u}{\partial \nu}(x_0)$ actually exists so take the problem statement to mean,

$$\limsup_{h \to 0^+} \frac{u(x_0) - u(x_0 - h\nu)}{h} < 0.$$

Hint: First use strong maximum principle to conclude that $u > 0$ in U or u is constant. If $u > 0$ in U try to show that $u(x) \geq c(|x - x_1|^{2-n} - r^{2-n})$ in $B(x_1,r) \setminus B(x_1,r/2)$ for some small $c > 0$.

Problem 6 (Jan 21)**.** Let U be a bounded domain of $\mathbb{R}^n$ with C^2 boundary, in particular it has an interior tangent ball at every boundary point. Using the result of the previous problem show that any two solutions of the Neumann problem

$$(N) \qquad \begin{cases} -\Delta u = f & \text{in} \quad U \\ \frac{\partial u}{\partial \nu} = h & \text{on} \quad \partial U \end{cases}$$

differ by a constant.

Problem 7 (Jan 21)**.** Let U be a bounded domain of $\mathbb{R}^n$ and $b : \overline{U} \to \mathbb{R}^n$ be continuous. Prove that there is at most one solution $u \in C^2(U) \cap C(\overline{U})$ of the Dirichlet problem,

$$\begin{cases} -\Delta u + b(x) \cdot \nabla u = f(x) & \text{in} \quad U \\ u = g(x) & \text{on} \quad \partial U. \end{cases}$$

Hint: Prove a weak maximum principle for solutions of $-\Delta u + b(x) \cdot \nabla u = 0$. You should try to follow the second method I used in lecture, first show the maximum principle for *strict subsolutions* satisfying $-\Delta u + b(x) \cdot \nabla u < 0$. Next, for non-strict subsolutions (e.g. solutions), you will need to make a perturbation like we did in class, using $v(x) = \delta |x|^2$ will not work anymore though so you will need to find a better function to perturb by. I recommend to look for a perturbing function $v(x_1)$ (as opposed to looking for something radial).

Problem 8 (Jan 25)**.** Evans, 2nd edition, Chapter 2, Problem 6.

Problem 9 (Jan 25)**.** Evans, 2nd edition, Chapter 2, Problem 10.

Problem 10 (Jan 25)**.** Suppose that u is harmonic in $\mathbb{R}^n$ and grows sub-quadratically:

$$\lim_{R \to \infty} \frac{1}{R^2} \sup_{B_R} |u(x)| = 0.$$

Show that u is linear on $\mathbb{R}^n$.

Problem 11 (Jan 25)**.** Show that the set

$$X = \{u \text{ harmonic on } B(0,1) \text{ and } |u| \leq 1\}$$

is compact in $C(K)$ for any $K \subset B(0,1)$ compact. Show the same for

$$Y = \{u \text{ harmonic and non-negative on } B(0,1) \text{ and } u(0) \leq 1\}.$$

Note: More precisely show that for any sequence $(u_n)_{n=1}^{\infty}$ in X (or Y) and any compact $K \subset B(0,1)$ the sequence u_n has a uniformly convergent subsequence and the limit is also harmonic. To find the subsequence you will apply the Arzela-Ascoli theorem, you will need to use information about harmonic functions to show that the assumptions of Arzela-Ascoli hold.

Problem 12 (Jan 27)**.** Consider the problem of minimizing the Dirichlet energy

$$I[v] = \int_U |Dv|^2 \, dx$$

over the admissible class

$$\mathcal{A} = \{v \in C^2(\overline{U}) : v|_{\partial U} = 0 \ \text{ and } \ \int_U |v|^2 \, dx = 1\}.$$

Show that if u satisfies

$$I[u] = \min_{v \in \mathcal{A}} I[v]$$

then u solves the *Dirichlet eigenvalue problem* with eigenvalue $\lambda_0 = \min_{v\in\mathcal{A}} I[v]$

$$\begin{cases} -\Delta u = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{1}$$

Show that if λ_1 is another eigenvalue of the Dirichlet Laplacian on U, i.e. there exists a non-trivial solution of (1) with $\lambda = \lambda_1$, then $\lambda_1 \geq \lambda_0$.

Hint: Note that perturbations $u+\varepsilon\varphi$ with $\varphi \in C_c^\infty(U)$ do not necessarily satisfy the L^2-integral constraint, instead try $(u+\varepsilon\varphi)/\|u+\varepsilon\varphi\|_{L^2(U)}$. [**EDIT**: Problem originally said C^1 test functions, that was not intended.]

Problem 13 (Jan 27)**.** Consider the energy

$$I[v] = \int_U \frac{1}{2}|Dv|^2 - fv \, dx$$

on the admissible class

$$\mathcal{A} = C^2(\overline{U}).$$

Show that if u satisfies

$$I[u] = \min_{v \in \mathcal{A}} I[v]$$

then u solves the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U. \end{cases}$$

Hint: First use test functions $\varphi \in C_c^\infty(U)$ (compactly supported in U), then use general test functions in $C^\infty(\overline{U})$. [**EDIT**: Problem originally said C^1 test functions, that was not intended.]

Problem 14 (Feb 2)**.** Evans, 2nd edition, Chapter 2, Problem 9.

Problem 15 (Feb 2)**.** ~~Evans, 2nd edition, Chapter 2, Problem 10.~~ [**EDIT**: Mistakenly repeated problem.]

Problem 16 (Feb 2)**.** Evans, 2nd edition, Chapter 2, Problem 11.

Problem 17 (Feb 7)**.** Let U a bounded domain in $\mathbb{R}^n$ and $\alpha \in (0,1)$. Suppose that $u \in C^2(U) \cap C(\overline{U})$ solves

$$\begin{cases} -\Delta u = |u|^\alpha & \text{in } U \\ u(x) = 0 & \text{on } \partial U \end{cases}$$

Show that

$$\sup_U |u| \leq C$$

where the constant C depends only on n, α and $\operatorname{diam}(U)$.

Hint: Evans Chapter 2, problem 6.

Problem 18 (Feb 7)**.** This problem considers the stationary Schrödinger operators $L = -\Delta + V(x)$ with zero Dirichlet data on a bounded domain U

$$\begin{cases} -\Delta u + V(x)u = 0 & \text{in } \ U \\ u = 0 & \text{on } \ \partial U \end{cases} \tag{2}$$

(a) Show by example (of domain and $V(x)$) that the maximum principle can fail for the above PDE.
Hint: For the sake of making an example just consider dimension $n = 1$ and an interval, think of eigenfunctions.
(b) Show that if $\sup |V(x)| \leq M$ there exists $\delta(M)$ so that if
$$\operatorname{diam}(U) \leq \delta$$
then the only solution $u \in C^2(U) \cap C(\overline{U})$ of (2) is $u \equiv 0$.
Hint: Similar idea to the proof of Evans Chapter 2, problem 6. Think of proving a bound for solutions of the Poisson equation $-\Delta u = f(x)$ in U with $u = 0$ on ∂U and then applying here.
(c) Show that if $\sup |V(x)| \leq M$ there exists $\delta(M)$ so that if
$$U \subset \{0 < x_1 < \delta\}$$
then the only solution $u \in C^2(U) \cap C(\overline{U})$ of (2) is $u \equiv 0$.

Problem 19 (Feb 7)**.** Show that if $U \subset \mathbb{R}^n$ is a connected *unbounded* domain and $u \in C^2(U) \cap C(\overline{U})$ solves

$$\begin{cases} -\Delta u = 0 & \text{in } \ U \\ u = 0 & \text{on } \ \partial U \\ \lim_{|x| \to \infty} u(x) = 0 \end{cases}$$

then $u = 0$.

Problem 20 (Feb 7)**.** Show that if $U \subset \mathbb{R}^n$ is a bounded domain (connected) and $u, v \in C^2(U) \cap C(\overline{U})$ solve

$$-\Delta w + |Dw|^2 = 0 \ \text{ in } \ U$$

with

$$u \leq v \ \text{ on } \ \partial U$$

then $u \leq v$ in U. This is called a *comparison principle* which is a generalization of the maximum principle appropriate for nonlinear equations.

Hint: First assume that u is a strict subsolution $-\Delta u + |Du|^2 < 0$ and prove the result. Use the first and second derivative tests at the point where $\max(v - u)$ is obtained. Then for the case $-\Delta u + |Du|^2 = 0$ try perturbing $u(x) + \varepsilon\phi(x)$ to make u a strict subsolution with $\phi(x) = e^{-cx_1}$.

Problem 21 (Feb 7)**.** Consider the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in } B(0,1) \setminus \{0\} \\ u(x) = 0 & \text{on } \partial B(0,1) \\ u(0) = -1. \end{cases} \tag{3}$$

Show that if we define the maximal subsolution of this problem

$$u(x) := \sup\{v(x) : \ v \text{ is a subsolution of } (3)\}$$

then $u \equiv 0$.

Hint: Try using the fundamental solution to create a sequence of subsolutions which converge to 0.

Problem 22 (Feb 14)**.** Evans, 2nd edition, Chapter 2, Problem 14.

Problem 23 (Feb 14)**.** Evans, 2nd edition, Chapter 2, Problem 15.

Problem 24 (Feb 14)**.** Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function with

$$\lim_{x \to -\infty} g(x) = a \ \text{ and } \ \lim_{x \to +\infty} g(x) = b.$$

Let $u(x,t)$ be the solution of the heat equation on $\mathbb{R} \times (0, \infty)$ given by,

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t) g(y) \, dy \ \text{ with } \ \Phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

Show that for every $R > 0$

$$\sup_{|x| \leq R} |u(x,t) - \frac{a+b}{2}| \to 0 \ \text{ as } \ t \to \infty.$$

Problem 25 (Feb 17)**.** Show the following bounds on moments of the heat kernel

$$\int_{\mathbb{R}^n} \Phi(y,t)|y|^p dy \leq C t^{p/2}$$

for a constant C depending on p and the dimension.

Problem 26 (Feb 17)**.** Let $g : \mathbb{R}^n \to \mathbb{R}$ be bounded and α-Hölder continuous for some $\alpha \in (0,1]$

$$[g]_{C^\alpha} := \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x-y|^\alpha} < +\infty.$$

Let u be the solution of the heat equation with initial data g, i.e. $u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\ dy$ where Φ is the heat kernel. Show that there is a constant C depending only on the dimension n so that

$$\sup_{x,t} |\partial_t u| + \sum_{i,j} \sup_{x,t} |\partial^2_{x_i x_j} u| \leq C[g]_{C^\alpha} t^{\frac{\alpha}{2}-1}.$$

We can use this estimate in the proof of Duhamel's formula to reduce the regularity requirement on $f(x,t)$ to just uniformly bounded and uniformly Hölder continuous in x, briefly explain why this works.

Hint: The result of the previous problem is useful. Notice also that $\int_{\mathbb{R}^n} \Phi(x-y,t) dy = 1$ for all t so $0 = \partial^2_{x_i x_j} \int_{\mathbb{R}^n} \Phi(x-y,t) dy = -\int_{\mathbb{R}^n} \partial^2_{y_i y_j} \Phi(x-y,t)\ dy$ and also $0 = \partial_t \int_{\mathbb{R}^n} \Phi(x-y,t) dy$.

Problem 27 (Feb 23)**.** Consider the heat equation IBVP set in a bounded domain U

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U \times (0,\infty) \\ u(x,0) = g(x) \end{cases}$$

with $g(x) \geq 0$. We will consider both Dirichlet and Neumann boundary conditions:

(a) Show that under the Dirichlet condition $u = 0$ on $\partial U \times (0,\infty)$ we have

$$\frac{d}{dt} \int_U u(x,t)\ dx \leq 0.$$

(b) Show that under the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial U \times (0,\infty)$

$$\frac{d}{dt} \int_U u(x,t)\ dx = 0.$$

(c) Assume that $u(x,t) \to u_\infty(x)$ as $t \to \infty$ in a sufficiently strong sense to justify that (1) u_∞ is a stationary solution of the heat equation in U i.e. it is harmonic $-\Delta u_\infty = 0$, (2) the Dirichlet/Neumann boundary conditions on ∂U are still satisfied by u_∞. In each case (Dirichlet/Neumann) identify u_∞ in terms of g.

Problem 28 (March 15)**.** Suppose that $\rho(x,t)$ is a smooth solution of the reaction-diffusion equation

$$\rho_t = \rho_{xx} + \rho(1-\rho) \;\text{ for }\; t > 0 \;\text{ with }\; 0 \leq \rho(x,0) \leq 1$$

with $\rho(x,t)$ is 1-periodic on $\mathbb{R}$ for all $t \geq 0$. Show that $0 \leq \rho(x,t) \leq 1$ for all $t > 0$.

Problem 29 (March 15)**.** Evans, 2nd edition, Chapter 2, Problem 19(d).

Problem 30 (March 15)**.** Evans, 2nd edition, Chapter 2, Problem 23.

Problem 31 (March 15)**.** Evans, 2nd edition, Chapter 2, Problem 24.

Problem 32 (March 16)**.** Let $A : \mathbb{R}^n \to S_n$ where S_n is the space of real symmetric $n \times n$ matrices. Suppose that $0 \leq A(x) \leq \Lambda$ in the sense of matrices, i.e. $\Lambda I - A$ and A are non-negative definite. Show that there is at most one smooth solution of the wave type equation,

$$\begin{cases} u_{tt} - \nabla \cdot (A(x)\nabla u) = f(x,t) & \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u(x,0) = \phi(x), \; u_t(x,0) = \psi(x) & \text{in} \quad \mathbb{R}^n. \end{cases}$$

Hint: You should show a finite speed of propagation property as we did in class. One of the key elements is to select a backwards light cone with a large enough propagation speed. Note that it is not possible to do this by looking at the energy on all of $\mathbb{R}^n$ (why?). You may find it useful to prove the following inequality for all vectors $v, w \in \mathbb{R}^n$ (which uses just that $A(x) \geq 0$),

$$2\langle A(x)v, w\rangle \leq \langle A(x)v, v\rangle + \langle A(x)w, w\rangle.$$

Problem 33 (March 18)**.** In one dimension one can prove from D'Alembert's formula that the solution of the wave equation with intial data $(u, u_t) = (\phi, 0)$ satisfies the bound

$$\sup_{x,t} |u(x,t)| \leq \sup_{x} |\phi(x)|. \tag{4}$$

Consider the initial value problem for the wave equation in $\mathbb{R}^3$:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad \mathbb{R}^3 \times (0,\infty) \\ u = \phi,\ u_t = \psi & \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{5}$$

We will show that a uniform in time estimate on the supremum norm (similar to (4) in dimension 1) cannot hold for solutions of the wave equation in $\mathbb{R}^3$. Heuristically speaking, initially small (in supremum norm) disturbances can be arranged to concentrate at some future time resulting in very large local oscillations.

(a) Let us first consider a situation with ϕ and ψ smooth and supported in $B(0, 1)$. Prove that $u(x, t)$ is supported in the annulus $B(0, t + 1) \setminus \overline{B}(0, t - 1)$ and,

$$\sup_x |u(x,t)| \leq C\frac{1}{t^2}(\sup_x |\phi(x)| + t \sup_x |D\phi(x)| + t \sup_x |\psi(x)|),$$

for some constant C.

(b) Show that if u is a solution of the wave equation for $t \in (0, \infty)$ and $T > 0$ then

$$v(x,t) = u(x, T + t) + u(x, T - t) \ \text{ solves the wave equation in } \ \mathbb{R}^3 \times (0,T)$$

with $v(x, 0) = 2u(x, T)$ and $v_t(x, 0) = 0$.

(c) For every $\varepsilon > 0$ small and $M > 0$ large give an example of a compactly supported initial data for the wave equation in $\mathbb{R}^3$ so that $\sup_x |\phi|, \sup_x |D\phi| \leq \varepsilon$, $\psi(x) = 0$ but there is a positive time $T(\varepsilon)$ so that $\sup_x |u(x, T)| \geq M$.

Remark: This problem is an example of a more general principle. For a time reversible equation, if certain initial data leads to decay in time of some norm of the solution, then also there must be initial data which leads to growth in time of that norm.

Problem 34 (March 30)**.** Evans, 2nd edition, Chapter 3, Problem 5.

Problem 35 (March 31)**.** Consider the transport equation

$$u_t + a(x,t) \cdot \nabla u = 0.$$

Prove that if u is a C^1 solution and $\beta : \mathbb{R} \to \mathbb{R}$ is an arbitrary C^1 function then $\beta(u(x, t))$ is a solution as well.

Problem 36 (March 31)**.** Find the characteristics for the problem,

$$u_t = \frac{1}{2}((u_x)^2 + x^2)$$

with the initial condition $u(x,0) = x$. The solution will not be defined for $|t| \geq \pi/2$. Explain that from the behavior of the characteristics.

Problem 37 (April 4)**.** Consider the solution of Burger's equation with smooth initial data u_0,

$$\begin{cases} u_t + uu_x = 0 & \text{in} \quad \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}. \end{cases}$$

Suppose that u_0' is bounded in $\mathbb{R}$.

(1) Compute, in terms of u_0, the first time t_* (possibly $+\infty$) when characteristics cross.
(2) Derive a necessary and sufficient condition on u_0 so that $t_* = +\infty$ and therefore Burger's equation has a smooth solution for all $t > 0$.

Problem 38 (April 4)**.** Consider the following scalar conservation law

$$\begin{cases} u_t + f(u)_x = 0 & \text{in} \quad \mathbb{R} \times (0,\infty) \\ u(x,0) = -x & \text{in} \quad \mathbb{R}. \end{cases}$$

Show that if $f''(z) \geq \theta > 0$ for all $z \in \mathbb{R}$ then $\inf_{x \in \mathbb{R}} \partial_x u(x,t) \to -\infty$ in a finite time.

Problem 39 (April 4)**.** (Evans 2nd edition, Chapter 3 problem 19) Assume that $f(0) = 0$ and u is a continuous integral solution of

$$\begin{cases} u_t + f(u)_x = 0 & \text{in} \quad \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}, \end{cases}$$

and u has compact support in $\mathbb{R} \times [0,T]$ for each $T > 0$, prove that,

$$\int_{\mathbb{R}} u(x,t)\ dx = \int_{\mathbb{R}} u_0(x)\ dx$$

for all $t > 0$.

Problem 40 (April 7)**.** The general existence and uniqueness theory for scalar conservation laws of the form,

$$u_t + F(u)_x = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty) \tag{6}$$

is based on the idea of *entropy solutions*. Let η be a smooth convex function, we call this an *entropy*, define the *entropy flux* q by,

$$q(u) := \int_0^u \eta'(v) F'(v)\, dv.$$

The pair (η, q) is called an *entropy pair*.

(a) Show that if u^ε is a smooth solution of the viscous approximation to the conservation law,

$$u^\varepsilon_t + F(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} \quad \text{in} \quad \mathbb{R} \times (0, \infty)$$

and (η, q) is an entropy pair then,

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \leq \varepsilon \eta(u^\varepsilon)_{xx} \quad \text{in} \quad \mathbb{R} \times (0, \infty). \tag{7}$$

Since we expect the physically relevant solutions of (6) to arise as limits of the viscous approximation we expect that (7) should hold in some form for the (correct) solutions of (6). This motivates the definition of an *entropy solution*:

Definition 1. Say $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is an *entropy solution* of (6) if u is a weak solution of (6) and for every entropy pair (η, q) and every *non-negative* test function φ compactly supported in $\mathbb{R} \times (0, \infty)$,

$$\int_{\mathbb{R} \times (0,\infty)} \eta(u)\varphi_t + q(u)\varphi_x \; dxdt \geq 0. \tag{8}$$

(b) Show that if u is an entropy solution of (6) in a region V of space-time and u is smooth on either side of a smooth parametrized curve (a shock) $C = \{(\gamma(t), t) : t \in I \subset \mathbb{R}\}$ with u, u_t, and u_x uniformly continuous in the regions V_ℓ and V_r to the left and right of C then the shock satisfies the Lax entropy condition,

$$f'(u_\ell(\gamma(t), t)) \geq \gamma'(t) \geq f'(u_r(\gamma(t), t)) \quad \text{for all} \quad t \in I.$$

Here u_ℓ and u_r are the left and right limits of u along C respectively.

Hint: First show that (8) implies a kind of Rankine-Hugoniot condition for $\eta(u)$. Then choose a good entropy/entropy flux pair.

(c) Show that if u is an entropy solution of (6) s.t. $u(x,t)$ has compact support in x for each $t > 0$, with initial data $u(x, 0) = u_0(x)$ then for every $p \geq 1$,

$$\int_{\mathbb{R}} |u(x,t)|^p \; dx \leq \int_{\mathbb{R}} |u_0(x)|^p \; dx \quad \text{for all} \quad t > 0.$$

Give an example of a weak solution of Burger's equation for which this inequality does not hold.

Problem 41 (April 12)**.** Evans, 2nd edition, Chapter 3, Problem 20.

Problem 42 (April 12)**.** Evans, 2nd edition, Chapter 3, Problem 9.

Problem 43 (April 12)**.** Evans, 2nd edition, Chapter 3, Problem 10.

Problem 44 (April 12)**.** Evans, 2nd edition, Chapter 3, Problem 13.

Problem 45 (April 12)**.** Evans, 2nd edition, Chapter 3, Problem 14.

The remaining problems are optional and do not need to be turned in. They are just some of my recommendations for learning the material from the last week of the class.

Problem 46. Evans, 2nd edition, Chapter 5, Problem 7.

Problem 47. Evans, 2nd edition, Chapter 6, Problem 4.

Problem 48. Evans, 2nd edition, Chapter 6, Problem 5.