HOMEWORK PROBLEMS

Each problem shows the date assigned. Problems are due one week after the assignment date, each problem will have a separate gradescope assignment which you can use to help keep track of due dates. See the most up to date syllabus for more details about late submissions etc.

Problem 1 (Aug 25). Consider the chemical reaction system
\[
\begin{aligned}
\dot{n}_{AB} &= k_1 n_A n_B - k_2 n_{AB} \\
\dot{n}_A &= \dot{n}_B = k_2 n_{AB} - k_1 n_A n_B.
\end{aligned}
\]
(1) Show that \(n_A - n_B, n_{AB} + n_A, n_{AB} + n_B,\) and \(n_{AB} + \frac{1}{2}(n_A + n_B)\) are all invariant quantities under the evolution (i.e. they are constant functions of \(t\)).
(2) Use this to rewrite the initial value problem as a single equation for \(x(t) = n_A(t)\) with parameters \(\alpha = n_A(0) - n_B(0)\) and \(\beta = n_{AB}(0) + n_A(0)\).
(3) Use phase line analysis to determine the long time behavior of \(x(t)\).

Note: You can (and should) use the physical assumption that all \(k_1, k_2, n_A(0), n_B(0), n_{AB}(0)\) are non-negative.

Problem 2 (Aug 25). Show that the ODE
\[
\ddot{x} + \mu \dot{x} - f(x) = 0
\]
cannot have any non-constant time periodic solution if \(\mu > 0\). Hint: Use the energy.

Problem 3 (Aug 25). Prove the following generalization of the Grönwall inequality. If \(a(t), b(t)\) and \(c(t)\) are continuous functions of \(t\) with \(c(t) > 0\) and
\[
x(t) \leq a(t) + b(t) \int_0^t c(s)x(s) \, ds
\]
then
\[
x(t) \leq a(t) + b(t) \left( \int_0^t c(s)a(s)\exp\left( \int_s^t c(u)b(u) \, du \right) \, ds \right).
\]
Note: \(a, b\) and \(c\) are not assumed to be differentiable.
Problem 4 (Aug 30). A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is said to have modulus of continuity $\rho$ if
\[ |g(x) - g(y)| \leq \rho(|x - y|) \]
where $\rho$ is a non-negative, continuous, monotone increasing function on $[0, \infty)$ with $\rho(0) = 0$. For example Lipschitz continuous functions have modulus of continuity $\rho(r) = Lr$.

A modulus of continuity $\rho$ is said to satisfy the Osgood condition if
\[ \int_0^1 \frac{1}{\rho(r)} dr = +\infty. \]
For example the Lipschitz modulus $\rho(r) = Lr$ satisfies this, but the Hölder modulus $\rho(r) = C r^\alpha$ for $\alpha \in (0, 1)$ does not. The modulus $\rho(r) = Lr(1 + |\log(r)|)$ is weaker than Lipschitz but still satisfies the Osgood property.

Suppose that $\rho$ is an Osgood modulus and $f(t, x)$ has
\[ |f(t, x) - f(t, y)| \leq \rho(|x - y|) \quad \text{for all } t \text{ and } x, y \in \mathbb{R}^n. \]
Show that the ODE IVP
\[ \dot{x} = f(t, x) \quad \text{with } x(0) = x_0 \]
has at most one solution.

In the specific case $\rho(r) = Lr(1 + |\log(r)|)$ write out explicitly an estimate on the difference $|x(t) - y(t)|$ in terms of $|x_0 - y_0|$ for two solutions analogous to the estimate
\[ |x(t) - y(t)| \leq e^{Lt} |x_0 - y_0| \]
which we derived in the case of a Lipschitz modulus.

\textbf{Hint:} Follow the uniqueness argument we did in the Lipschitz case. At some point in that calculation you will find that you need to prove a variant of Grönwall’s inequality. Follow the proof of Grönwall’s inequality to do this.
**Problem 5** (Sept 1). Suppose that $U$ is a smooth domain in $\mathbb{R}^n$ which can be written in the sub-level set form

$$U = \{ x : g(x) > 0 \} \quad \text{and} \quad \partial U = \{ x : g(x) = 0 \}$$

where $g$ is a smooth real valued function on $\mathbb{R}^n$ with $|\nabla g(x)| \neq 0$ for $x \in \partial U$. Let $f(t, x)$ be a vector field which is Lipschitz continuous in $x$ with constant $L$ and

$$f(t, x) \cdot \nu(x) = 0 \quad \text{for all} \quad x \in \partial U$$

where $\nu(x)$ is the outward normal direction to $\partial U$ at $x$. Consider ODE evolution associated with the vector field $f$

$$\dot{x} = f(t, x) \quad \text{and} \quad x(t_0) = x_0$$

with flow map $\phi_t(x_0, t_0) = x(t)$. Show the following:

(a) $\partial U$ is an invariant set under the flow $\phi_t$.
(b) $U$ is an invariant set under the flow $\phi_t$.
(c) What if we assumed only that $f(t, x) \cdot \nu(x) < 0$ for all $t$ and $x \in \partial U$.

What could you conclude about the invariance of $U$ and $\partial U$? (May be helpful to draw a picture)

**Note:** The special form of $U$ is only designed to make computations easier. Recall that the gradient of $g$ points normal to its level sets and $\nu(x) = -\nabla g(x)/|\nabla g(x)|$ for $x \in \partial U$.

**Hint:** For part (a) try computing $\frac{d}{dt}g(\phi_t(x))$ and then showing that, for initial data in $\partial U$, $|\frac{d}{dt}g(\phi_t(x))| \leq A|g(\phi_t(x))|$ as long as $g(\phi_t(x))$ is sufficiently small (for an appropriate constant $A$). Then apply Grönwall. I will record an additional hint for part (a) which you will be able to find on canvas. For part (c) and initial data inside of $U$ try looking at the first time $t_*$ that $g(\phi_t(x)) = 0$, since $g$ is positive inside of $U$ the time derivative at this time $t_*$ of $g(\phi_t(x))$ will be non-positive, but what does the equation say?

**Problem 6** (Sept 3). Use an ODE argument to prove that

$$\det(e^A) = e^{\text{tr}(A)}.$$

**Problem 7** (Sept 3). Suppose all eigenvalues of $A$ have negative real part. Show that every solution of

$$\dot{x} = Ax$$

converges to 0 as $t \to \infty$. What is the rate of convergence? Does it depend on initial data? How?

**Hint:** First do the case when all eigenvalues are distinct so that $A$ is diagonalizable over $\mathbb{C}$. Then for general $A$, and you may use this result without proof, for any $\varepsilon > 0$ there is a matrix $B$ with $\|B - A\|_{op} \leq \varepsilon$ and all eigenvalues of $B$ are distinct. Write $\dot{x} = Ax = Bx + (A - B)x$ and treat the second term as a forcing term applying the Duhamel / variation of parameters formula. Then apply the result of problem 3.
Problem 8 (Sept 3). Suppose all eigenvalues of $A$ have negative real part and $g(t) \in \mathbb{R}^n$ is continuous and $\lim_{t \to \infty} |g(t)| = 0$. Show that every solution of

$$\dot{x} = Ax + g(t)$$

converges to 0 as $t \to \infty$. What if $\lim_{t \to \infty} g(t) = g_0$?

[Teschl, *Ordinary Differential Equations and Dynamical Systems*, Problem 3.16]

Problem 9 (Sept 8). Write the second order system

$$\ddot{u} + 2\dot{u} + u = 0$$

as a $2 \times 2$ first order system $\dot{x} = Ax$. Compute the matrix exponential $e^{At}$. Use this to find a general formula for the solution with initial data $u(0) = u_0$ and $\dot{u}(0) = u_1$.

[Sideris, *Ordinary Differential Equations and Dynamical Systems*, p. 18, via course of A. Treibergs]

Problem 10 (Sept 8). Let $A$ be a real $2 \times 2$ matrix. Then the eigenvalues can be expressed in terms of the determinant $D = \det(A)$ and the trace $T = \text{tr}(A)$. In particular the pair $(T, D)$ can take all possible values in $\mathbb{R}^2$ as $A$ ranges over all $2 \times 2$ real matrices. Draw a diagram splitting the $(T, D)$ plane into regions where each possible behavior occurs (stable node, unstable node, stable spiral, unstable spiral, saddle, center).

[Teschl, *Ordinary Differential Equations and Dynamical Systems*, Problem 3.14]