HOMEWORK PROBLEMS

Each problem shows the date assigned. Problems are due one week after the assignment date, each problem will have a separate gradescope assignment which you can use to help keep track of due dates. See the most up to date syllabus for more details about late submissions etc.

**Problem 1** (Aug 25). Consider the chemical reaction system

\[
\begin{align*}
\dot{n}_{AB} &= k_1 n_A n_B - k_2 n_{AB} \\
\dot{n}_A &= \dot{n}_B = k_2 n_{AB} - k_1 n_A n_B.
\end{align*}
\]

1. Show that \( n_A - n_B, n_{AB} + n_A, n_{AB} + n_B, \text{ and } n_{AB} + \frac{1}{2}(n_A + n_B) \) are all invariant quantities under the evolution (i.e. they are constant functions of \( t \)).
2. Use this to rewrite the initial value problem as a single equation for \( x(t) = n_A(t) \) with parameters \( \alpha = n_A(0) - n_B(0) \) and \( \beta = n_{AB}(0) + n_A(0) \).
3. Use phase line analysis to determine the long time behavior of \( x(t) \).

**Note:** You can (and should) use the physical assumption that all \( k_1, k_2, n_A(0), n_B(0), n_{AB}(0) \) are non-negative.

**Problem 2** (Aug 25). Show that the ODE

\[\ddot{x} + \mu \dot{x} - f(x) = 0\]

cannot have any non-constant time periodic solution if \( \mu > 0 \). Hint: Use the energy.

**Problem 3** (Aug 25). Prove the following generalization of the Grönwall inequality. If \( a(t), b(t) \) and \( c(t) \) are continuous functions of \( t \) with \( c(t) > 0 \) and

\[x(t) \leq a(t) + b(t) \int_0^t c(s)x(s) \, ds\]

then

\[x(t) \leq a(t) + b(t) \left( \int_0^t c(s)a(s) \exp \left( \int_s^t c(u)b(u) \, du \right) \, ds \right).

**Note:** \( a, b \) and \( c \) are not assumed to be differentiable.
Problem 4 (Aug 30). A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is said to have modulus of continuity $\rho$ if
\[ |g(x) - g(y)| \leq \rho(|x - y|) \]
where $\rho$ is a non-negative, continuous, monotone increasing function on $[0, \infty)$ with $\rho(0) = 0$. For example Lipschitz continuous functions have modulus of continuity $\rho(r) = Lr$.

A modulus of continuity $\rho$ is said to satisfy the Osgood condition if
\[ \int_0^1 \frac{1}{\rho(r)} \, dr = +\infty. \]
For example the Lipschitz modulus $\rho(r) = Lr$ satisfies this, but the Hölder modulus $\rho(r) = Cr^\alpha$ for $\alpha \in (0, 1)$ does not. The modulus $\rho(r) = Lr(1 + |\log(r)|)$ is weaker than Lipschitz but still satisfies the Osgood property.

Suppose that $\rho$ is an Osgood modulus and $f(t, x)$ has
\[ |f(t, x) - f(t, y)| \leq \rho(|x - y|) \]
for all $t$ and $x, y \in \mathbb{R}^n$.

Show that the ODE IVP
\[ \dot{x} = f(t, x) \quad \text{with} \quad x(0) = x_0 \]
has at most one solution.

In the specific case $\rho(r) = Lr(1 + |\log(r)|)$ write out explicitly an estimate on the difference $|x(t) - y(t)|$ in terms of $|x_0 - y_0|$ for two solutions analogous to the estimate
\[ |x(t) - y(t)| \leq e^{Lt} |x_0 - y_0| \]
which we derived in the case of a Lipschitz modulus.

**Hint:** Follow the uniqueness argument we did in the Lipschitz case. At some point in that calculation you will find that you need to prove a variant of Grönwall’s inequality. Follow the proof of Grönwall’s inequality to do this.
Problem 5 (Sept 1). Suppose that $U$ is a smooth bounded domain in $\mathbb{R}^n$ which can be written in the sub-level set form

$$U = \{ x : g(x) > 0 \} \quad \text{and} \quad \partial U = \{ x : g(x) = 0 \}$$

where $g$ is a smooth real valued function on $\mathbb{R}^n$ with $|\nabla g(x)| \neq 0$ for $x \in \partial U$. Let $f(t, x)$ be a vector field which is Lipschitz continuous in $x$ with constant $L$ and

$$f(t, x) \cdot \nu(x) = 0 \quad \text{for all} \quad x \in \partial U$$

where $\nu(x)$ is the outward normal direction to $\partial U$ at $x$. Consider ODE evolution associated with the vector field $f$

$$\dot{x} = f(t, x) \quad \text{and} \quad x(t_0) = x_0$$

with flow map $\phi_t(x_0, t_0) = x(t)$. Show the following:

(a) $\partial U$ is an invariant set under the flow $\phi_t$.
(b) $U$ is an invariant set under the flow $\phi_t$.
(c) What if we assumed only that $f(t, x) \cdot \nu(x) < 0$ for all $t$ and $x \in \partial U$.

What could you conclude about the invariance of $U$ and $\partial U$? (May be helpful to draw a picture)

Note: The special form of $U$ is only designed to make computations easier. Recall that the gradient of $g$ points normal to its level sets and $\nu(x) = -\nabla g(x)/|\nabla g(x)|$ for $x \in \partial U$.

Hint: For part (a) try computing $\frac{d}{dt} g(\phi_t(x))$ and then showing that, for initial data in $\partial U$, $|\frac{d}{dt} g(\phi_t(x))| \leq A \|g(\phi_t(x))\|$ as long as $g(\phi_t(x))$ is sufficiently small (for an appropriate constant $A$). Then apply Grönwall. I will record an additional hint for part (a) which you will be able to find on canvas. For part (c) and initial data inside of $U$ try looking at the first time $t_*$ that $g(\phi_t(x)) = 0$, since $g$ is positive inside of $U$ the time derivative at this time $t_*$ of $g(\phi_t(x))$ will be non-positive, but what does the equation say?

Problem 6 (Sept 3). Use an ODE argument to prove that

$$\det(e^A) = e^{\text{tr}(A)}.$$

Problem 7 (Sept 3). Suppose all eigenvalues of $A$ have negative real part. Show that every solution of

$$\dot{x} = Ax$$

converges to 0 as $t \to \infty$. What is the rate of convergence? Does it depend on initial data? How?

Hint: First do the case when all eigenvalues are distinct so that $A$ is diagonalizable over $\mathbb{C}$. Then for general $A$, and you may use this result without proof, for any $\varepsilon > 0$ there is a matrix $B$ with $\|B - A\|_{op} \leq \varepsilon$ and all eigenvalues of $B$ are distinct. Write $\dot{x} = Ax = Bx + (A - B)x$ and treat the second term as a forcing term applying the Duhamel / variation of parameters formula. Then apply the result of problem 3.
**Problem 8** (Sept 3). Suppose all eigenvalues of $A$ have negative real part and $g(t) \in \mathbb{R}^n$ is continuous and $\lim_{t \to \infty} |g(t)| = 0$. Show that every solution of
\[
\dot{x} = Ax + g(t)
\]
converges to 0 as $t \to \infty$. What if $\lim_{t \to \infty} g(t) = g_0$?

[Teschl, *Ordinary Differential Equations and Dynamical Systems*, Problem 3.16]

**Problem 9** (Sept 8). Write the second order system
\[
\ddot{u} + 2\dot{u} + u = 0
\]
as a $2 \times 2$ first order system $\dot{x} = Ax$. Compute the matrix exponential $e^{At}$. Use this to find a general formula for the solution with initial data $u(0) = u_0$ and $\dot{u}(0) = u_1$.

[Sideris, *Ordinary Differential Equations and Dynamical Systems*, p. 18, via course of A. Treibergs]

**Problem 10** (Sept 8). Let $A$ be a real $2 \times 2$ matrix. Then the eigenvalues can be expressed in terms of the determinant $D = \det(A)$ and the trace $T = \text{tr}(A)$. In particular the pair $(T, D)$ can take all possible values in $\mathbb{R}^2$ as $A$ ranges over all $2 \times 2$ real matrices. Draw a diagram splitting the $(T, D)$ plane into regions where each possible behavior occurs (stable node, unstable node, stable spiral, unstable spiral, saddle, center).

[Teschl, *Ordinary Differential Equations and Dynamical Systems*, Problem 3.14]

**Problem 11** (Sept 17). Show that if $|\varepsilon|$ is sufficiently small then all solutions of Mathieu’s equation are bounded
\[
\ddot{x} + (1 + \varepsilon \sin(3t))x = 0.
\]

**Note:** You can quote the result of Problem 14 below.

[U. Utah PhD Preliminary Examination in Differential Equations, January 2004, via Treibergs course Fall 2017]

**Problem 12** (Sept 13). Show that the ODE
\[
\dddot{x} - (\cos^2 t)\dot{x} + (\sin^2 t)x = 0
\]
does not have any bounded fundamental set of solutions. **Note:** There was previously a typo here omitting the dot in $\dot{x}$ so double check that you have the equation written down correctly as it is now.

[U. Utah PhD Preliminary Examination in Differential Equations, Fall 2017]
Problem 13 (Sept 20). (Resonant forcing) Consider the ODE

\[ \dot{x} = A(t)x + b(t) \]

with \( A(t) \) and \( b(t) \) both \( T \)-periodic and let \( \Phi(t) \) be the principal fundamental solution of the homogeneous problem \( \dot{\Phi} = A(t)\Phi \) and \( \Phi(0) = I \). Suppose that 1 is an eigenvalue of \( \Phi(T) \) with multiplicity 1 (both algebraic and geometric). Show that there exists a solution \( x \) of (1) of the form

\[ x(t) = y(t) + tz(t) \]

where \( z \) is some \( T \)-periodic solution of \( \dot{z} = A(t)z \) and \( y(t) \) is \( T \)-periodic.

**Hint:** Start by looking for the equation solved by \( y \).

Problem 14 (Sept 20). (Continuous dependence on parameters) Consider the \( \mu \in [a,b] \)-parametrized family of ODE systems

\[ \dot{x} = f(t,x,\mu). \]

Suppose that \( f \) satisfies the Lipschitz conditions

\[ |f(t,x,\mu) - f(t,y,\mu)| \leq L|x - y| \]

for all \( x, y \in \mathbb{R}^n \), \( t \in \mathbb{R} \) and \( \mu \in [a,b] \) and

\[ |f(t,x,\mu) - f(t,\mu',\mu')| \leq M(1 + |x|)|\mu - \mu'| \]

for all \( x \in \mathbb{R}^n \), \( t \in \mathbb{R} \), and \( \mu, \mu' \in [a,b] \). Show that the flow map \( \phi_t(x,\mu) \) is Lipschitz continuous with respect to \( \mu \) for each \( t > 0 \) and find an upper bound for the Lipschitz constant.

Problem 15 (Sept 22). (Dissipative system) Let \( p(t) > 0 \) be a smooth function which is periodic of least period \( T > 0 \). Show that every solution approaches zero as \( t \to \infty \) for the ODE

\[ \ddot{x} + p(t)\dot{x} + x = 0. \]

[U. Utah PhD Preliminary Examination in Differential Equations, Fall 2014]

Problem 16 (Sept 22). Show that for small \( |\varepsilon| \), there is a \( 2\pi \)-periodic solution near zero to the inhomogeneous equation

\[ \ddot{y} + 5y = \varepsilon \sin^3 t. \]

[U. Utah PhD Preliminary Examination in Differential Equations, Fall 2014]

Problem 17 (Sept 24). Use the Poincaré-Bendixson Theorem to show that the system

\[ \begin{cases} \dot{x} = x - y - x^3 \\ \dot{y} = x + y - y^3 \end{cases} \]

has a periodic orbit in the annular region \( A = \{1 < |(x,y)| < \sqrt{2}\} \).

**Hint:** First figure out where are the fixed points of the system. Then show that the region \( A \) is forward invariant by applying an appropriate part
of Problem 5. This will help you to apply Poincaré-Bendixson and get the desired conclusion.

[P. Bressloff’s MATH 6410 course, Fall 2015]

Problem 18 (Sept 24). Consider Selkov’s model of glycolysis, the process whereby living cells break down sugar, where \( x \) and \( y \) are concentrations of ADP and F6P. Show that it has a nonconstant periodic solution.

\[
\dot{x} = -x + \frac{y}{10} + x^2 y \\
\dot{y} = \frac{1}{2} - \frac{y}{10} - x^2 y.
\]

Hint: Similar ideas to previous problem, show the polygon with vertices \((0, 0), (0, 5), \left(\frac{1}{2}, 5\right)\) and \(\left(\frac{11}{2}, 0\right)\) is forward invariant.

[U. Utah PhD Preliminary Examination in Differential Equations, Fall 2014]

Problem 19 (Sept 29). Given the potential \( U(x) \) sketched here:

[diagram of potential U(x)]

make a careful sketch of the phase plane for the Hamiltonian system associated with

\[
H(p, x) = \frac{1}{2} |p|^2 + U(x)
\]

including fixed points, heteroclinic/homoclinic orbits, etc.

[Perko, *Differential Equations and Dynamical Systems*, p. 179]

Problem 20 (Oct 1). Consider the Lorenz system

\[
\dot{x} = \sigma (y - x) \\
\dot{y} = \rho x - y - xz \\
\dot{z} = xy - \beta z
\]

with \( \sigma > 0, \rho > 0 \) and \( \beta > 0 \).

(a) Show that this system is invariant under the transformation \((x, y, z, t) \mapsto (-x, -y, z, t)\).
(b) Show that the $z$-axis is invariant under the flow and it consists of three trajectories.

(c) Show that this system has equilibrium points at the origin and at $(\pm \sqrt{\beta (\rho - 1)}, \pm \sqrt{\beta (\rho - 1)}, \rho - 1)$ for $\rho > 1$. For $\rho > 1$ show that there is a one-dimensional unstable manifold $W_u(0)$ at the origin.

(d) For $\rho \in (0, 1)$ use the Lyapunov function $V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma z^2$ to show that the origin is globally stable; i.e. for $\rho \in (0, 1)$ the origin is the $\omega_+$-limit set of every trajectory of the system.

[Perko, Differential Equations and Dynamical Systems, Section 3.2 Problem 6]

**Problem 21** (Oct 4). (Degeneracy of center manifolds) Consider the system

$$\dot{x} = -x \quad \text{and} \quad \dot{y} = y^2.$$ 

Find all smooth invariant manifolds on the form $\{(h(y), y) : y \in \mathbb{R}\}$ which are tangent to the center subspace $E_c$ of the linearization at 0.

[Teschl, Ordinary Differential Equations and Dynamical Systems, Problem 9.9]

**Problem 22** (Oct 6). (Differentiability with respect to initial data) Suppose that $f(x)$ is $C^1$ show that the flow map $\phi_t(x_0)$ associated with the ODE

$$\dot{x} = f(x)$$

is differentiable with respect to $x_0$ and for all $1 \leq j \leq n$

$$\frac{d}{dt} \partial_j \phi_t(x_0) = Df(\phi_t(x_0))\partial_j \phi_t(x_0).$$

Hint: Fix $x_0$ and $j$ and first find the equation solved by the difference quotients $w_h(t) = \phi_t(x_0 + he_j) - \phi_t(x_0)$. You can use fundamental theorem of calculus to get a linear ODE for $w_h(t)$ (it will still be nonlinear in terms of $\phi_t(x_0)$). In the integral form compare this ODE with the (integral form of) equation $\dot{w} = Df(\phi_t(x_0))w(t)$ show that the limit as $h \to 0$ of $w_h(t)$ exists and solves this limiting equation.

**Problem 23** (Oct 20). Fill in the gaps from lecture to show that if $f(x)$ is a $C^2$ vector field, 0 is a periodic point for the flow of $f$ with period $T$, $\Sigma$ is a transversal manifold to the flow at 0, $P_\Sigma$ is the corresponding Poincaré map and

$$\|DP_\Sigma(0)\|_{op} < 1$$

then, for every $x_0$ in a sufficiently small neighborhood of 0, $\omega_+(x_0) = \Gamma_0$ (the periodic orbit through 0).

**Problem 24** (Oct 20). Suppose that $f$ is a vector field on $\mathbb{R}^2$ and $x(t)$ is a periodic orbit with period $T$ show that if

$$\int_0^T (\nabla \cdot f)(x(t)) \, dt < 0$$

then the orbit is stable.
**Problem 25** (Oct 20). Investigate the system
\[ \dot{x} = -y + (\mu + \sigma(x^2 + y^2))x, \quad \dot{y} = x + (\mu + \sigma(x^2 + y^2))y \]
as a function of the parameter \( \mu \) for \( \sigma = 1 \) and for \( \sigma = -1 \). Find the stable and unstable sets of the periodic orbits and stationary points. Hint: use polar coordinates.

[ Teschl, *Ordinary Differential Equations and Dynamical Systems*, Problem 12.1 ]

**Problem 26** (Oct 25). Study the nonlinear systems
\[ \dot{x} = \epsilon x(1-x) \sin^2 t \]
and
\[ \dot{x} = \epsilon (x \sin^2 t - x^2/2) \]
by the method of averaging. Write out the first asymptotic expansion for solutions of each (can write implicitly in terms of the averaged solution), and explicitly write out the expansion for the stationary solutions of the averaged equation. What do you notice about their solutions?

[Guckenheimer and Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, problem 4.2.3]

**Problem 27** (Oct 27). Consider the nonlinear equation
\[ \ddot{x} + \epsilon \dot{x} (x^2 - 1) + x = 0 \quad \text{with} \quad 0 < \epsilon \ll 1. \]

1. Use the method of multiple scales to show that this has an asymptotic series solution of the form
   \[ x(t) = R(\epsilon t) \cos(t + \Theta(\epsilon t)) + O(\epsilon) \]
   with
   \[ \Theta_{\tau} = 0 \quad \text{and} \quad R_{\tau} = \frac{1}{8} R(4 - R^2) \]
   where \( \tau = \epsilon t \)

2. Derive the solution
   \[ R(\tau) = \frac{2}{(1 + a_0 e^{-\epsilon \tau})^{1/2}} \]
   and establish that there is a stable periodic orbit.

[U. Utah PhD Preliminary Examination in Differential Equations, Fall 2015]

**Problem 28** (Oct 27). For the perturbed harmonic oscillator \( \ddot{x} + x = \epsilon x \), the natural frequency is “corrected” at first order in the perturbation parameter by \( \omega(\epsilon) = 1 - \epsilon \). What is the first order correction if the perturbation is \( \epsilon x^2 \) or \( \epsilon x^3 \)? What about \( \epsilon x^n \)?

[Chicone, *Ordinary Differential Equations with Applications*, p. 433]
**Problem 29** (Nov 8). Consider the forced Duffing equation
\[ \ddot{x} + x = \varepsilon [\gamma \cos \omega t - \delta \dot{x} - \alpha x^3] \]
first consider the case \( \omega = 1 \) and use the method of multiple scales (or method of averaging) to find the asymptotic expansion
\[ x(t) = A(\varepsilon t) \cos t + B(\varepsilon t) \sin t + O(\varepsilon) \]
identifying the differential equations solved by \( A(\tau) \) and \( B(\tau) \). You may also write the expansion in amplitude-phase form if you prefer. Next consider the near resonant case \( \omega = 1 + \varepsilon \Omega \) and do the same thing. (You do not need to solve the resulting ODE’s for \( A \) and \( B \), we may look into them more deeply later).

**Problem 30** (Nov 10). Consider the differential operator
\[ \mathcal{L} = \frac{d^2}{dx^2} + \omega^2 \]
with the domain
\[ D(\mathcal{L}) = \{ f, \mathcal{L}f \in L^2([0,1]) : f(0) = f'(0) = 0 \} \]
find an inverse operator \( \mathcal{I} \) on \( D(\mathcal{L}) \) for \( \mathcal{L} \) and show that \( \mathcal{I} \) is compact. Hint: Duhamel.

**Problem 31** (Nov 10). Find the adjoint operator and associated adjoint boundary conditions for each of the following examples
1. \( L = \frac{d^2}{dx^2} + a(x) \frac{d}{dx} + b(x) \) with the boundary conditions \( u(0) = u'(1) \) and \( u(1) = u'(0) \).
2. \( L = -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \) with the boundary conditions \( u(0) = u(1) \) and \( u'(0) = u'(1) \).

**Problem 32** (Nov 16). Suppose that \( \{ \phi_k \}_{k=1}^{\infty} \) is an orthonormal set in \( L^2([0,1]) \) such that the series expansion
\[ \sum_{k=1}^{\infty} \langle \phi_k, u \rangle_{L^2([0,1])} \phi_k(x) \] converges pointwise on \([0,1]\) for all \( u \in C^\infty([0,1]) \). Show that if \( \{ \phi_k \}_{k=1}^{\infty} \) is an orthonormal basis of \( L^2([0,1]) \) then
\[ \sum_{k=1}^{\infty} \phi_k(x) \phi_k(y) = \delta(x - y) \] in the sense of distributions.

The convergence of this series is in the sense of distributions which means: for each \( u \in C^\infty([0,1]) \)
\[ \int_0^1 \left[ \sum_{k=1}^{N} \phi_k(x) \phi_k(y) \right] u(y) \, dy \to \langle \delta_x, u \rangle \text{ as } N \to \infty. \]
Remark: This is actually an if and only if and gives a condition for an orthonormal set to be complete.
Problem 33 (Nov 18). Find the Green’s function for each of the following boundary value problems and then give a corresponding representation formula for the solution

(a) The Dirichlet problem

\[ u'' + \omega^2 u = f(x) \quad \text{in } [0,1] \quad \text{with } u(0) = u(1) = 0. \]

For which values of \( \omega \) does the standard Green’s function fail to exist?

(b) The periodic boundary conditions problem

\[ u'' + \omega^2 u = f(x) \quad \text{in } [0,1] \quad \text{with } u(0) = u(1) \text{ and } u'(0) = u'(1). \]

This problem has a zero eigenvalue for all \( \omega \) due to the constant solution \( L1 = 0 \). What is the condition on \( f \) for this problem to have a solution? Find the modified Green’s function for the values of \( \omega \) for which it is possible.

(c) The mixed Dirichlet-Neumann problem

\[ u'' + \frac{3}{2x} u' - \frac{3}{2x^2} u = f(x) \quad \text{in } [0,1] \quad \text{with } u(0) = 0, \quad u'(1) = 0. \]

When looking for solutions of \( Lu = 0 \) try a guess of the form \( u(x) = x^p \).

Problem 34 (Nov 19). Consider the following PDE initial boundary value problem in \( \mathbb{R}^2 \)

\[ \partial_t^2 u - (\Delta + V(|x|))u = 0 \quad \text{in } B_1 \quad \text{with } u(t,x) = 0 \quad \text{on } \partial B_1 \quad \text{and } u(0,x) = u_0. \]

Use the method of separation of variables in polar coordinates, assuming that

\[ u(t,x_1,x_2) = T(t)R(r)\Theta(\theta) \]

to find three associated regular Sturm-Liouville eigenvalue problems. Precisely state those problems including the boundary conditions, check that the operators/boundary conditions are self-adjoint (with a weighted \( L^2 \) inner product if necessary).

Hint: The Laplace operator in polar coordinates is

\[ \Delta u = \partial_r^2 u + \frac{1}{r} \partial_r u + \frac{1}{r^2} \partial_{\theta \theta} u \]