MATH 6220 HOMEWORK 2

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Final version. If you do not have access to the textbook for problems let me know and I can help.

Problems:

1. Homotopy.
   (a) Prove that any star-shaped region is simply connected. A region Ω is called star-shaped if there is a point $z_0 \in \Omega$ so that for any $z \in \Omega$ the line segment $[z_0, z] \subset \Omega$.
   (b) Show that $\mathbb{C} \setminus (-\infty, 0]$ is simply connected.
   (c) Give two non-homotopic simple closed curves in $\mathbb{D} \setminus \{0\}$ which contain the point $\frac{1}{2}$.
2. Stein & Shakarchi, Chapter 2, **Exercises** 5, 6, 7, 9
3. Show that if $f : \mathbb{C} \to \mathbb{C}$ holomorphic has at most polynomial growth:
$$|f(z)| \leq A + B|z|^{\alpha} \quad \text{for all} \quad z \in \mathbb{C}$$
for some $A, B, \alpha \in [0, \infty)$ then f is a polynomial.
4. Prove that if f is entire and $\lim_{|z|\to\infty} |f(z)| = +\infty$ then f is a polynomial. [**HINT**: Look at $g(w) = \frac{1}{f(1/w)}$, use the result of problem 5, and look at the power series expansion around $w = 0$.]
5. Show that if $f : \mathbb{D} \to \mathbb{C}$ is continuous and holomorphic in $\mathbb{D} \setminus \{1/2\}$ then f is holomorphic in $\mathbb{D}$. [**EDIT**: To be clear: f is continuous in the entire disc $\mathbb{D}$ including $1/2$.]
6. Show that if f is holomorphic in a domain Ω and $D_r(z_0) \subset \Omega$ then
$$f(z_0) = \frac{1}{2\pi r} \int_{C_r(z_0)} f(\zeta) d|\zeta|.$$
Also show that
$$f(z_0) = \frac{1}{\pi r^2} \int_{D_r(z_0)} f(\zeta) dA(\zeta)$$
where the integral dA is the standard Lebesgue area integral in $\mathbb{C}$ as identified with $\mathbb{R}^2$. [**EDIT**: Typos corrected]
7. Show that the family
$$\mathcal{F} = \{f : \Omega \to \mathbb{C} |\ f \text{ holomorphic on } \Omega \text{ and } |f| \leq M\}$$
is compact as a subset of $C(K) = \{f : K \to \mathbb{C} : \text{continuous}\}$ for any K compact in Ω. [**HINT/EDIT**: Statement is unclearly phrased instead show the following: Show that every sequence in $\mathcal{F}$ has a

subsequence which converges uniformly on K a function in $C(K)$ which is holomorphic in $\text{int}(K)$.]

8. Show that the family

$$\mathcal{F} = \{f : \Omega \to \mathbb{C}|\ f \text{ holomorphic on } \Omega \text{ and } \int_\Omega |f|^2 dA \leq M\}$$

is compact as a subset of $C(K) = \{f : K \to \mathbb{C} : \text{continuous}\}$ for any K compact in Ω. [**HINT/EDIT**: Statement is unclearly phrased instead show the following: Show that every sequence in $\mathcal{F}$ has a subsequence which converges uniformly on K a function in $C(K)$ which is holomorphic in $\text{int}(K)$.]

9. Given an open set $\Omega \subset \mathbb{C}$ show that Ω has an exhaustion by compact sets, i.e. a family K_n of compact subsets of Ω so that $K_1 \subset K_2 \subset \cdots$ and $\Omega = \cup_{n=1}^{\infty} K_n$. (Try to make a simple construction)

10. Say that a sequence $f_n \in C(\Omega)$ converges *locally uniformly* in Ω to a function f if

$$\sup_K |f_n - f| \to 0 \quad \text{as} \quad n \to \infty$$

for all K compact in Ω. Find a metric for this topology. **Hint:** $\rho(a, b) = \frac{d(a,b)}{1+d(a,b)}$ is a metric when d is.

11. Show that the family of holomorphic function $\mathcal{F}$ from problem 7 is compact in the topology of local uniform convergence on Ω.

12. Stein & Shakarchi, Chapter 2, **Exercise** 10.