COURSE OUTLINE MATH 6420

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1. Basics

General form of a PDE $F(D^k u, \ldots, u, x) = 0$ in U with boundary conditions. Order of equation, linear (also semilinear, quasilinear and fully nonlinear). Systems of equations.

1.1. Equations and applications.

Transport, Laplace, Heat, Wave, Schrödinger. Types of equations (parabolic, elliptic, hyperbolic/dispersive) and energy concepts unifying.

1.2. Types of questions we ask about PDE.

Well-posedness (existence, uniqueness (+stability)), Qualitative properties (regularity, special solutions, scaling limits).

1.3. Conservation / balance laws.

Field $u(x,t)$ compute $\frac{d}{dt}\int_\Omega u(x,t)\,dx$. 1. Write the balance law with flux F and source s, 2. divergence theorem and Ω arbitrary, 3. Relate flux (and maybe source term too) to original field u to "close" the system via constitutive relation, this is some more addition of the physics of the scenario. Continuity equation - flux bu. Heat equation - energy conservation $e = \rho c u$ (density, specific heat, temperature), heat flux $Q = \kappa \nabla u$ (thermal conductivity), source term. Wave equation - u is displacement, momentum conservation ρu_t, flux $F = -k\nabla u$ (k is bulk modulus).

2. Transport equation

2.1. constant coefficient.

Start with constant $b \in \mathbb{R}^n$

$$u_t + b \cdot Du = 0 \ \text{ in } \ \mathbb{R}^n \times (0,\infty).$$

Notice that this is vanishing of a space-time directional derivative, define

$$z(s) = u(x + sb, t + s)$$

then

$$\dot{z} = u_t + b \cdot Du = 0$$

so u is constant on lines parallel to $(b, 1)$ in $\mathbb{R}^n \times \mathbb{R}$. Draw picture. Now initial value problem

$$\begin{cases} u_t + b \cdot Du = 0 & \text{in } \ \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x). \end{cases}$$

Note that $(x+sb, t+s)$ hits the $t=0$ plane at $s=-t$ and $y = x - tb$ so

$$u(x,t) = g(x-tb)$$

solves. Solutions translates in direction b. Derivation shows that any sufficiently regular solution must be given by this formula, and if $g \in C^1$ then this formula solves. On the other hand, even if g is not C^1 (it can even be discontinuous!) the formula provides a (the only) reasonable candidate solution. This is first example of weak solution.

2.2. **non-constant coefficient.** Now take $b : \mathbb{R}^n \to \mathbb{R}^n$ globally Lipschitz

$$u_t + b(x) \cdot Du = 0 \text{ in } \mathbb{R}^n \times (0,\infty).$$

Look for some trajectories $X(t)$ so that $X(0) = x_0$

$$0 = \frac{d}{ds} u(X(t), t) = u_t + \dot{X}(t) \cdot Du$$

can satisfy if

$$\dot{X}(t) = b(X(t)).$$

This is ODE flow! Draw a picture. So

$$u(x,t) = g(x_0, t)$$

x_0 is the initial data for ODE so that $x = X(t; x_0) = \phi_t(x_0)$ so $x_0 = \phi_t^{-1}(x) = \phi_{-t}(x)$ i.e.

$$u(x,t) = g(\phi_{-t}(x), t)$$

2.3. **non-homogeneous.** Try the same idea but $\dot{z}$ is non-zero now.

3. Laplace / Poisson equation

3.1. **Fundamental solution.** (Look for solution sharing the scaling invariances of the equation). We will look for

$$u(x) = v(r)$$

note

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}|x|^{-1} 2x_i = x_i / r$$

so

$$u_{x_i} = v'(r)\frac{x_i}{r} \quad \text{And} \quad u_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)(\frac{1}{r} - \frac{x_i^2}{r^3})$$

so

$$\Delta u = v''(r) + \frac{(n-1)}{r} v'(r)$$

so

$$(\log|v|)' = \frac{1-n}{r}.$$

Fundamental solution is defined

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}.$$

Note $|D\Phi| \leq C|x|^{1-n}$ and $|D^2\Phi| \leq C|x|^{-n}$. Note that the Hessian has a singularity at 0 which is apparently not integrable.

3.2. **Solving Poisson equation in whole space.** Observe that translations and linear combinations of Φ are also Laplace solutions (except at singularities!). Define

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\ dy$$

for $f \in C^2_c(\mathbb{R}^n)$ (regularity not so important). Then $u \in C^2$ and $-\Delta u = f$.

Reminders about convolutions and divergence theorem!

Proof: 1. C^2 apply difference quotients to f and use uniform convergence of difference quotients to derivatives. 2. Apply Laplacian, split integral into $B(0,\varepsilon)$ and $\mathbb{R}^n \setminus B(0,\varepsilon)$, estimate inner integral, integrate by parts in outer integral estimate boundary term. 3. integrate by parts again in remaining exterior term, use harmonicity, and finally compute last boundary term.

Interpret as $-\Delta\Phi = \delta_0$!

3.3. **The mean value formula.** If u harmonic then

$$u(x) = \fint_{\partial B(x,r)} u dS = \fint_{B(x,r)} u dy$$

proof: Define

$$\phi(r) = \fint_{\partial B(x,r)} u(y)dS(y) = \fint_{\partial B(0,1)} u(x+rz)dS(z)$$

take the derivative

$$\phi'(r) = \fint_{\partial B(0,1)} Du(x+rz)\cdot z dS(z)$$

go back to original y variable and note that this is integral of $\frac{\partial u}{\partial \nu}$, then apply divergence theorem. Evaluate

$$\int_{B(x,r)} u(y)\ dy = \int_0^r \int_{\partial B(x,s)} u(y)dS(y)ds.$$

Converse: If u is $C^2(U)$ and satisfies mean value property then u is harmonic. Suppose $\Delta u \neq 0$ somewhere in U, then use previous computation to find a contradiction of MVP.

3.4. **Mollification.** The proof uses the idea of *mollification.* The function

$$\eta(x) = \begin{cases} ce^{-\frac{1}{1-|x|^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

is $C^\infty_c(\mathbb{R}^n)$. Choose the normalizing constant c so that $\int_{\mathbb{R}^n} \eta(x)\ dx = 1$.

We define a recaled family $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$, this is called a family of *mollifiers*. If we take

$$\eta_\varepsilon \star f(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) f(y)\ dy$$

then this family of functions is C^∞ and we can typically show

$$\eta_\varepsilon \star f \to f \quad \text{as} \quad \varepsilon \to 0$$

with the notion of convergence depending on what functional space f lies in. For example if $f \in L^p(\mathbb{R}^n)$ convergence will hold in L^p, if $f \in C_c(\mathbb{R}^n)$ convergence will hold in uniform norm.

The proof always has the following basic structure (doing the $C_c(\mathbb{R}^n)$ case as an example) using that $\int \eta_\varepsilon(x-y)\ dy = 1$

$$\eta_\varepsilon \star f(x) - f(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)(f(y) - f(x))$$

and then using that $\eta_\varepsilon(x-y) = 0$ for $|y-x| \geq \varepsilon$ and for $|y-x| \leq \varepsilon$ we use the uniform continuity of f, $|f(y) - f(x)| \leq \omega(|y-x|) \leq \omega(\varepsilon)$ so

$$|\eta_\varepsilon \star f(x) - f(x)| \leq \omega(\varepsilon) \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\ dy = \omega(\varepsilon)$$

etc etc. Make remark on derivative bound $|D^k \eta_\varepsilon \star f| \leq C\varepsilon^{-k}\|f\|_{L^1(B_{2\varepsilon})}$

3.5. **Regularity of harmonic functions.** If $u \in C(U)$ satisfies the mean value property for every ball $B(x,r) \subset U$ then u is C^∞ in U and, in particular, satisfies $\Delta u = 0$ in U.

Now application to the MVP implies smooth: Let u as in the statement, the mollification $\eta_\varepsilon \star u_\varepsilon$, defined in $U_\varepsilon = \{x \in U : d(x, \partial U) > \varepsilon\}$, is smooth and

$$\begin{aligned}
\eta_\varepsilon \star u(x) = \int \eta_\varepsilon(x-y)u(y)\ dy \\
&= \frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} \eta\left(\frac{|x-y|}{\varepsilon}\right)u(y)\ dy \\
&= \frac{1}{\varepsilon^n}\int_0^\varepsilon \int_{\partial B(0,r)} \eta\left(\frac{r}{\varepsilon}\right)u(x+z)\ dS(z)\ dr \\
&= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)n\alpha(n)r^{n-1}u(x)\ dr \\
&= u(x)\int_{B(0,\varepsilon)} \eta_\varepsilon(y)\ dy = u(x)
\end{aligned}$$

So the mollification $\eta_\varepsilon \star u(x) = u(x)$ for all $x \in U_\varepsilon$, i.e. u is C^∞ in U_ε. Since $\varepsilon > 0$ was arbitrary u is smooth in U.

Make remark about bound on derivatives depending on distance to the boundary.

3.6. **Interior regularity estimates.** If u is harmonic in $B(0,r)$ then

$$|D^\alpha u(0)| \leq \frac{C}{r^{|\alpha|}} \fint_{B(0,r)} |u| \, dx.$$

Proof: Differentiate solid ball MVP, integrate by parts, bound by sup on $B(0,r/2)$. Bound sup on $B(0,r/2)$ by L^1 average on $B(0,r)$ by MVP again. Induction for general $|\alpha|$.

Give minor spiel about why regularity is important: weak solution to classical solution, quantitative converge when consider sequences of solutions, sequences of PDES (i.e. in a numerical method).

3.7. **Liouville.** Bounded harmonic functions in $\mathbb{R}^n$ are constant.

Proof: apply Lipschitz estimate on large ball.

Remark that regularity implies Liouville, but, in some sense, Liouville also implies regularity.

Corollary: Bounded solutions of Poisson equation in $\mathbb{R}^n$ unique up to constants.

3.8. **Harnack Inequality.** Positive harmonic function in $B(0,3)$ then

$$\sup_{B(0,1)} u \leq C \inf_{B(0,1)} u.$$

Similar result for u positive and harmonic in U and $K \subset U$ compact with $C = C(K,n)$.

Proof: Let $x, y \in B(0,1)$, then $B(0,3) \supset B(x,2) \supset B(y,1)$. Mean value property

$$u(x) = \fint_{B(x,2)} u(z) \, dz \geq \frac{1}{|B(x,2)|} \fint_{B(y,1)} u(z) \, dz = \frac{1}{2^n} u(y).$$

Notes: Harnack inequality by itself implies Hölder regularity, implies Liouville, etc etc. Another expression of the interioa averaging affect of the Laplace equation.

3.9. **Maximum principle.** Suppose $u \in C(\overline{U}) \cap C^2(U)$. Weak maximum principle:

$$\max_{\overline{U}} u = \max_{\partial U} u$$

Strong maximum principle: If U is connected and there is $x_0 \in U$ with $u(x_0) = \max_{\overline{U}} u$ then u is constant in U.

Same results for minima.

Proof 1: Use mean value property and show that $\{x \in U : u(x) = M\}$ is open and closed in U.

Introduce subharmonic/superharmonic, and strictly sub/super -harmonic.

Proof 2: Show that strictly subharmonic attains max on ∂U. Show that subharmonic can be perturbed to strictly subharmonic.

Show that uniqueness for Dirichlet problem is a corollary of weak maximum principle. Show that strict positivity of non-negative harmonic functions is a corollary of strong-maximum principle.

3.10. **Boundary value problems.** State Dirichlet and Neumann problems. Mention mixed / Robin problems. Recall uniqueness via maximum principle.

3.11. **Energy methods.** Start with uniqueness: suppose two solution u, v and call $w = u - v$

$$0 = \int_U -w\Delta w \; dx = \int_U |Dw|^2 \; dx.$$

Dirichlet's principle: energy functional

$$I[u] = \int_U \frac{1}{2}|Du|^2 - uf \; dx$$

defined on the admissible class

$$\mathcal{A} = \{u \in C^2(\overline{U}) : u|_{\partial U} = g\}$$

If $u \in C^2(\overline{U})$ solves the Dirichlet problem then

$$u = \min_{w \in \mathcal{A}} I[w]$$

and conversely if u minimizes I over $\mathcal{A}$ then u solves the Dirichlet problem.

proof: 1. Let's compute the derivative of I in direction $\varphi \in C_c^1(U)$

$$\langle DI[u], \varphi\rangle := \frac{d}{dt}\Big|_{t=0} I[u + t\varphi] = \cdots = \int_U (-\Delta u - f)\varphi.$$

So as an element of the dual of $C_0(U)$ $DI[u]$ is represented by $-\Delta u - f$.

2. Convexity: since $p \mapsto \frac{1}{2}|p|^2$ is convex on $\mathbb{R}^n$

$$I[(1-t)u+tv] = \int_U \frac{1}{2}|(1-t)Du+tDv|^2-(1-t)uf-tvf \; dx \leq (1-t)I[u]+tI[v].$$

Note that this also implies local version of convexity (I lies above it's linearization) by sending $t \to 0$: rewrite above inequality as

$$tI[v] \geq tI[u] + (I[u + t(v-u)] - I[u])$$

then divide by t and send $t \to 0$ to find

$$I[v] \geq I[u] + \langle DI[u], v - u\rangle$$

if $v - u$ is zero on ∂U then our previous formula for $DI[u]$ applies.

3. Now suppose u solves the Dirichlet problem. Let $w \in \mathcal{A}$. Since $w - u$ is zero on ∂U we know $\langle DI[u], w - u\rangle = 0$ so convexity implies

$$I[w] \geq I[u] + DI[u](w - u) = I[u].$$

i.e. u minimizes I over $\mathcal{A}$.

4. Now suppose u minimizes I over $\mathcal{A}$. Then

$$i(t) = I[u + t\varphi]$$

has a minimum at $t = 0$ for any $\varphi \in C_c^1(U)$. Then $i'(0) = 0$ and so

$$0 = i'(0) = DI[u](\varphi) = \int_U (-\Delta u - f)\varphi \; dx.$$

Since $\varphi \in C_c^1(U)$ was arbitrary

$$-\Delta u = f \quad \text{pointwise in } U.$$

3.12. **Connection with stochastic processes.** Consider an iid random walk process $X_t(x)$ started at x on a domain $\Lambda \subset \mathbb{Z}^n$ which is killed when it exits Λ. When at site $x \in \Lambda$ the random walker chooses a neighbor $y \sim x$ uniformly at random and steps to y. If $y \notin \Lambda$ the process stops. More precisely

$$\mathbb{P}(X_{t+1} = y | X_t = x) = \frac{1}{2d}\mathbf{1}(y \sim x).$$

Call $\partial\Lambda$ to be the outer site boundary, $\tau(x)$ to be the first time that the process started at x exits Λ. Then $X_{\tau(x)}(x) \in \partial\Lambda$. The harmonic measure ω_x is the probability measure on $\partial\Lambda$ defined by

$$\omega_x(E) = \mathbb{P}(X_{\tau(x)}(x) \in E).$$

The function $u(x) = \omega_x(E)$ is discrete harmonic in Λ with Dirichlet data $u(x) = \mathbf{1}_E$ on $\partial\Lambda$. Boundary data is clear (explain), equation follows from the logic

$$\omega_x(E) = \mathbb{P}(X_{\tau(x)}(x) \in E) = \sum_{y \sim x} \mathbb{P}(X_{\tau(y)}(y) \in E)\mathbb{P}(X_1(x) = y) = \frac{1}{2d}\sum_{y \sim x} \mathbb{P}(X_{\tau(y)}(y) \in E).$$

So

$$\Delta_{\mathbb{Z}^d} u = \frac{1}{2d}\sum_{y \sim x}(u(y) - u(x)) = 0.$$

By linearity general Dirichlet data can be solved by

$$u(x) = \sum_{y \in \partial\Lambda} g(y)\omega_x(\{y\}).$$

Many of the things we have proved about harmonic functions are true of discrete harmonic functions as well in some form, although the proofs can be more difficult because we don't have "continuous calculus".

Now let's think about the continuum case. Consider a Brownian motion $B_t(x)$ started from $x \in \Omega$. As before we are interested in the distribution of the location of the Brownian motion at the exit time from the domain

$$\tau(x) = \inf\{t : B_t(x) \notin \Omega\}.$$

Given a subset $E \subset \partial\Omega$ we define harmonic measure (probability of Brownian motion started at x to exit Ω through E)

$$\omega_x(E) = \mathbb{P}(B_{\tau(x)}(x) \in E).$$

Turns out that $v(x) = \omega_x(E)$ solves the Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v(x) = \mathbf{1}_E(x) & \text{on } \partial\Omega. \end{cases}$$

Boundary data is easy. We check $\omega_x(E)$ is harmonic by checking the mean value property: let $x \in \Omega$ and $\partial B(x, r) \subset \Omega$, call σ to be the exit time from

$B(x,r)$ of the Brownian motion started at x. Note $\sigma < \tau$. Also note that by rotation invariance of Brownian motion $B_\sigma(x)$ is uniformly distributed on $\partial B(x,r)$. So, by the strong Markov property of Brownian motion,

$$v(x) = \fint_{\partial B(x,r)} v(y) dS(y).$$

Thus v is harmonic in U.

Again by linearity we can solve arbitrary Dirichlet problems

$$u(x) = \int_{\partial U} g(y) d\omega_x(y).$$

Examples: corner exit probability.

3.13. **Green's functions.** Looking for solution formula for

$$-\Delta u = f \quad \text{in} \quad U \quad \text{with} \quad u = g \quad \text{on} \quad \partial U$$

analogous to what we did for the whole space problem.

Start out with the fundamental solution $\Phi(x) = c_n |x|^{2-n}$. By Green's formula

$$\int_U u(y)\Delta_y \Phi(y-x) - \Phi(y-x)\Delta u(y) \, dy = \int_{\partial U} u\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu} \, dS(y)$$

and

$$\int_U u(y)\Delta_y \Phi(y-x) \, dy = -u(x)$$

so

$$u(x) = \int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial \nu} - u(y)\frac{\partial \Phi}{\partial \nu}(y-x) \, dS(y) + \int_U \Phi(y-x)(-\Delta u(y)) \, dy$$

This looks close to a representation formula for the solution of the Dirichlet problem because we know $-\Delta u(y) = f$ and we know $u|_{\partial \Omega} = g$. However we do not know $\frac{\partial u}{\partial \nu}$ on $\partial \Omega$. We would like to eliminate this term by choosing a variant of the fundamental solution which satisfies a zero Dirichlet boundary condition.

Define a corrector function $\phi^x(y)$ to solve

$$-\Delta \phi^x = 0 \quad \text{in} \quad U \quad \text{and} \quad \phi^x(y) = \Phi(y-x) \quad \text{on} \quad \partial U$$

(note relies on existence for Dirichlet problem OR an explicit construction) then we define the **Green's function**

$$G(x,y) = \Phi(y-x) - \phi^x(y).$$

Then $G(x,y) = 0$ for $y \in \partial U$.

Note: By comparison principle in the domain $U \setminus B(x,\varepsilon)$ for small $\varepsilon > 0$ the function $G(x,y)$ is non-negative in U.

Applying the previous Green's theorem argument to $G(x,y)$ instead of $\Phi(y-x)$ we find the formula

$$u(x) = -\int_{\partial U} u(y)\frac{\partial G}{\partial \nu}(x,y) \, dS(y) + \int_U G(x,y)(-\Delta u(y)) \, dy.$$

Note that given our previous discussion of harmonic measure

$$d\omega_x(y) = -\frac{\partial G}{\partial \nu}(x,y)\ dS(y).$$

We also define the **Poisson kernel** of the domain U, $P : U \times \partial U \to \mathbb{R}$

$$P(x,y) = -\frac{\partial G}{\partial \nu}(x,y).$$

So the general representation formula for the solution of the Dirichlet problem is

$$u(x) = \int_{\partial U} P(x,y) g(y)\ dS(y) + \int_U G(x,y) f(y)\ dy.$$

NOTE: With $f = 0$ we see that $P(x,y)$ is the Radon-Nikodym derivative of the harmonic measure $d\omega_x(y)$ with respect to surface measure $dS(y)$ on ∂U:

$$d\omega_x(y) = P(x,y) dS(y).$$

[Symmetry of Green's function] Fix $x \neq y$. Call

$$v(z) = G(x,z) \quad \text{and} \quad w(z) = G(y,z).$$

Then $\Delta v = \delta_x$ and $\Delta w = \delta_y$, and both are zero on ∂U. Applying Green's identity

$$w(x) - v(y) = \int_U w\Delta v - v\Delta w\ dx = \int_{\partial U} w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu}\ dx = 0$$

3.14. **Green's function for a half-space.** Method of images, consider

$$G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x})$$

where

$$\tilde{x} = (x_1, \ldots, x_{n-1}, -x_n).$$

Then

$$G_{y_n}(x,y) = -\frac{1}{n\alpha(n)}[\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y-\tilde{x}|^n}$$

and if $y \in \partial\mathbb{R}^n_+$ then

$$|y-x| = |y-\tilde{x}|$$

so

$$-\frac{\partial G}{\partial \nu}(x,y) = G_{y_n}(x,y) = \frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n}$$

This is the half-space Poisson kernel. We check that

$$u(x) = \frac{2x_n}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{g(y)}{|y-x|^n}\ dy$$

solves the Dirichlet problem in the half-space.

Theorem: If $g \in C_c(\partial\mathbb{R}^n_+)$ then u solves the Dirichlet problem.

proof: Note that for $y \in \partial\mathbb{R}^n_+$ the Poisson Kernel $P(x,y) = -G_{y_n}(x,y)$. Since, for fixed y, G is harmonic in the x variable in $\mathbb{R}^n \setminus \{y\}$. In particular, when $y \in \partial\mathbb{R}^n_+$, we have $P(x,y)$ harmonic in x in $\mathbb{R}^n_+$.

Need to check boundary values, typical approximate identities argument.

3.15. **Green's function in the ball.** Method of images again, using the *Kelvin inversion*. For $x \in \mathbb{R}^n \setminus \{0\}$ define the inversion $\tilde{x} = x/|x|^2$.

We are going to use an image charge at $\tilde{x}$

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x})).$$

Note that this is harmonic in $B(0,1)\setminus\{x\}$, we just need to check the boundary condition for $y \in \partial B(0,1)$.

There $|y| = 1$ and

$$|x|^2|y-\tilde{x}|^2 = |x|^2(|y|^2 - 2y\cdot\frac{x}{|x|^2} + \frac{1}{|x|^2}) = |x|^2 - 2y\cdot x + 1 = |x-y|^2.$$

Now we have the Poisson formula for solution of Dirichlet problem in $B(0,1)$

$$u(x) = \int_{\partial B(0,1)} -\frac{\partial G}{\partial \nu}(x,y)g(y)\ dS(y).$$

Need to find the kernel.

$$\Phi_{y_i}(y-x) = \frac{-1}{n\alpha(n)}\frac{y_i - x_i}{|x-y|^n}$$

$$[\Phi(|x|(y-\tilde{x}))]_{y_i} = \frac{-1}{n\alpha(n)}\frac{y_i|x|^2 - x_i}{(|x||y-\tilde{x}|)^n} = -\frac{1}{n\alpha(n)}\frac{y_i|x|^2 - x_i}{|x-y|^n}.$$

the last equality holds if $y \in \partial B(0,1)$.

Then

$$\frac{\partial G}{\partial \nu}(x,y) = \sum y_i G_{y_i}(x,y) = \frac{-1}{n\alpha(n)}\frac{1}{|x-y|^n}\sum y_i(y_i - x_i - y_i|x|^2 + x_i)$$

$$P(x,y) = -\frac{\partial G}{\partial \nu}(x,y) = \frac{1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n}$$

Poisson formula... etc. Poisson formula for $B(0,r)$

$$P(x,y) = \frac{r^2 - |x|^2}{n\alpha(n)r}\frac{1}{|x-y|^n} \quad \text{for} \quad x \in B(0,r),\ y \in \partial B(0,r).$$

Give formula state result about boundary continuity.

3.16. **Existence of solutions to the Dirichlet problem via Perron's method.** The energy minimization can be used to show existence of solutions to the Dirichlet problem, but it requires more sophisticated functional analysis. There is a more elementary avenue to existence via *Perron's method*. This is also known as the *method of sub and supersolutions*, and it generalizes well to equations satisfying a maximum principle.

It suffices to consider the homogeneous problem

$$-\Delta u = 0 \ \text{ in } \ U \ \text{ and } \ u = g \ \text{ on } \ \partial U.$$

For this proof we will assume that g is continuous. That assumption is more or less necessary for this approach.

Consider the class of *subsolutions* of the Dirichlet problem

$$S = \{v \in C(\overline{U}) : -\Delta v \leq 0 \text{ in } U \text{ and } v \leq g \text{ on } \partial U\}.$$

The meaning of $-\Delta v \leq 0$ *is that* v *satisfies the sub-mean-value-property for a* **sufficiently small ball around each point***, this does make sense for just continuous functions.* If we had a solution $u \in C^2(U) \cap C(\overline{U})$ then $u \in S$, but also by maximum principle $v \leq u$ for any $v \in S$. That is u would be the *maximal subsolution* of the Dirichlet problem. We will try to construct u by finding the maximal subsolution.

Lemma: The maximum of two subharmonic functions is subharmonic.

Lemma: Comparison between sub and supersolutions.

Define

$$u(x) = \max\{v(x) : v \in S\}.$$

The idea is to show that u solves the Dirichlet problem.

Note that the class of subsolutions is nonempty, $\min g$ is in it, and it is bounded from above, $\max g$ is above every element of S by comparison principle.

Harmonic lifts: Given a function $v \in S$ and a ball $B \subset U$ we define the harmonic lift

$$v_B(x) = \begin{cases} v(x) & x \notin B \\ \int_{\partial B} P_B(x,y)v(y)\ dy & x \in B. \end{cases}$$

where P_B is the (explicitly constructed) Poisson kernel for the ball B. Note that by our results on the Poisson kernel v is continuous in U and harmonic in B.

We also want to check the following: $v_B \geq v$, and v_B is still subharmonic in the entire U in particular $v_B \in S$.

$v_B \geq v$ follows from maximum principle in B.

Subharmonic: Given any $x \in U$ we need to check sub-mean value property on sufficiently small balls centered at x. Case 1: $x \in B$, then choose $r > 0$ small enough so that $B_r(x) \subset B$ and we have mean value property because v_B is harmonic there. Case 2: $x \in U \setminus B$, then sub mean value property follows from sub mean value property for v and that $v_B \geq v$.

Lemma: The maximal subsolution $u(x)$ defined above is harmonic in U.

Proof: Fix $x_0 \in U$ and let $v_n \in S$ be a sequence of subsolutions so that $v_n(x_0) \nearrow u(x_0)$. Without loss $\min g \leq v_n \leq \max g$, the lower inequality is guaranteed by taking $\max\{v_n, \min g\}$. Let $r > 0$ sufficiently small so that $B(x_0, r) \subset\subset U$. Call $w_n = (v_n)_{B(x_0,r)}$ the harmonic lifts of v_n in that ball. Then the w_n are uniformly bounded still between $\min g \leq w_n \leq \max g$. Since they are harmonic in $B(x_0, r)$ and uniformly bounded by homework problem 11 there is a subsequence (not relabeled) of w_n converging uniformly in $B(x_0, r/2)$ to some w which is harmonic in $B(x_0, r/2)$.

Our claim is that $w = u$ in $B(x_0, r/2)$, at the moment we just know $w(x_0) = u(x_0)$ (why?).

Let $x_1 \in B(x_0, r/2)$ and let $\tilde{v}_n$ be a sequence of subsolutions with $\tilde{v}_n(x_1) \to v_n(x_1)$. Then $z_n = \max\{v_n, \tilde{v}_n\}$ are also subsolutions with $z_n(x_j) \to u(x_j)$ for $j \in \{0, 1\}$ and we can perform the harmonic lift in $B(x_0, r)$ on z_n as well to get $\tilde{w}_n \geq w_n$. By the same arguments (up to a subsequence) the $\tilde{w}_n$ converge uniformly to a function $\tilde{w} \geq w$ which is harmonic in $B(x_0, r/2)$ with $\tilde{w}(x_0) = u(x_0)$ and $\tilde{w}(x_1) = u(x_1)$. However now we have a non-positive harmonic function $w - \tilde{w}$ in $B(x_0, r/2)$ with $w(x_0) - \tilde{w}(x_0) = 0$. By strong maximum principle $w \equiv \tilde{w}$ in $B(x_0, r/2)$, in particular $w(x_1) = \tilde{w}(x_1) = u(x_1)$.

Since $x_1 \in B(x_0, r/2)$ was arbitrary $w \equiv u$ in $B(x_0, r/2)$ meaning u is C^∞ and harmonic in $B(x_0, r/2)$. Since $x_0 \in U$ was arbitrary same for whole domain U.

Boundary barriers: We still need to check that u has the correct boundary conditions. For this we use a *barrier argument.*

Definition 1. A function $\phi : \overline{U} \to \mathbb{R}$ is a *barrier for the Dirichlet problem* at $x_0 \in \partial U$ if

- $\phi(x_0) = 0$
- $\phi(x) > 0$ for $x \in \overline{\Omega} \setminus \{x_0\}$
- ϕ is superharmonic in Ω
- ϕ is continuous in $\overline{\Omega}$.

Lemma 2. *If there is a barrier for the Dirichlet problem at $x_0 \in \partial U$ then the Perron's method solution u has*

$$\lim_{U \ni x \to x_0} u(x) = g(x_0).$$

In particular if U has boundary barriers at every boundary point then $u \in C(\overline{U})$ and $u|_{\partial U} = g$.

Proof: Let $\varepsilon > 0$, g is continuous so $|g(x) - g(x_0)| \leq \varepsilon$ in $B_\delta(x_0)$. Note $\phi > 0$ on $\overline{U} \setminus B_\delta(x_0)$ so call $m > 0$ to be the minimum on that compact set. Then define

$$\psi(x) = g(x_0) + \varepsilon + \sup_U |g(x) - g(x_0)| \frac{1}{m} \phi(x)$$

This barrier is harmonic in U, it is larger than $g(x_0) + \varepsilon \geq g(x)$ on $B_\delta(x_0)$ and on $\partial U \setminus B_\delta(x_0)$ it is larger than $g(x)$. Thus $\psi \geq g$ on ∂U, since ψ is harmonic in U it is a supersolution and so $u \geq \psi$. (For the subsolution case we would argue that $\tilde{\psi}$ is a subsolution so it is smaller than the maximal subsolution u).

Thus $\limsup_{x \to x_0} u(x) \leq \limsup_{x \to x_0} \psi(x) = g(x_0) + \varepsilon$. Then send $\varepsilon \to 0$. $\square$

Barrier construction relies on *exterior* domain regularity. For example if U has an exterior touching ball at x_0 then U has a barrier at x_0. If $B(x_1, r)$ is the exterior touching ball at x_0 take

$$\phi(x) = r^{2-n} - |x - x_1|^{2-n}$$

as the boundary barrier.

If U has an exterior cone at x_0 can also construct a boundary barrier in radial coordinates as

$$\phi(r,\theta) = \Theta(\theta) r^{\alpha}$$

for an appropriate choice of α depending on the cone and the dimension and Θ found by separation of variables.

3.17. **Review.**

- Fundamental solutions / Green's functions
- Maximum principle
- Interior regularization (MVP, elliptic estimates, Harnack, Liouville)
- Energy minimization principle
- Connection with random walks

4. The heat equation

4.1. **Fundamental solution.** As before we start by exploiting invariances of the equation to find a specific solution, in this case it is a space-time scaling invariance and spatial rotation invariance. If u solves heat equation then $\lambda^{\alpha} u(\lambda x, \lambda^2 t)$ solves as well. We could search for a scaling invariant solution $u(x,t) = \lambda^{\alpha} u(\lambda x, \lambda^2 t)$ for all $\lambda \in \mathbb{R}$, i.e. $u(x,t) = t^{-\alpha} u(\frac{x}{t^{1/2}}, 1)$.

We will attempt to derive the correct scaling by writing a bit more generally

$$u(x,t) = \frac{1}{t^{\alpha}} v(\frac{x}{t^{\beta}})$$

plugging into equation calling $y = xt^{-\beta}$

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot Dv(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0$$

we want an equation which depends only on y which requires $\alpha + 2\beta = \alpha + 1$ i.e. $\beta = 1/2$ and α is still free to be chosen. Then

$$\alpha v + \frac{1}{2} y \cdot Dv + \Delta v = 0$$

and now looking for radial solutions $v(y) = w(|y|)$

$$\alpha w + \frac{1}{2} r w' + r^{1-n}(r^{n-1} w')' = 0$$

with $r = |y|$ and $' = \frac{d}{dr}$. In order to make the first term a derivative we can choose $\alpha = \frac{n}{2}$ so

$$\frac{1}{2}(r^n w)' + (r^{n-1} w')' = 0$$

or

$$\frac{1}{2} r w + w' = a$$

if w and $w' \to 0$ as $r \to \infty$ then the constant of integration should be $a = 0$ so

$$w' = -\frac{1}{2} r w$$

which we can integrate

$$w(r) = be^{-\frac{r^2}{4}}$$

So we have found a solution of the heat equation

$$u(x,t) = \frac{b}{t^{n/2}} e^{-|x|^2/4t}$$

Fundamental solution:

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} & t > 0 \\ 0 & t < 0. \end{cases}$$

The choice of constant is for the property

$$\int_{\mathbb{R}^n} \Phi(x,t)dx = 1 \quad \text{for all } t > 0.$$

Formally speaking $\Phi(x,0) = \delta_0$.

4.2. **Initial value problem.** Solution of heat equation initial value problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \end{cases}$$

Lemma: 1. (Regularization) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$, 2. u solves heat equation, 3. (Initial data) If $g \in C(\mathbb{R}^n)$ then initial data is achieved continuously.

proof: 1. $\Phi(x,t)$ is smooth for $t > 0$. 2. Φ solves heat equation for $t > 0$. 3. Typical approximate identities argument.

Point out *infinite speed of propagation*.

4.3. **Non-homogeneous problem.**

$$\begin{cases} \partial_t u - \Delta u = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = 0 \end{cases}$$

Formal derivation via Duhamel's formula

$$\partial_t(e^{-\Delta t}u) = f$$

so

$$u(t) = e^{\Delta t}g + \int_0^t e^{\Delta(t-s)} f(s)\ ds$$

gives a ansatz for the solution

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\ dyds = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\ dyds.$$

Can use linearity to find solution with both initial data and inhomogeneity.

Lemma: 1. u is as regular as f allows e.g. if $f \in C^{2,1}$ then $u \in C^{2,1}$ (actually there is improvement but it is a bit more technical), 2. u solves inhomogeneous heat equation. 3. Initial data 0 achieved continuously.

proof: 1. Differentiate under the integral putting derivatives on f. 3. Just do L^∞ bound on the convolution.

2. Put derivatives on f initially, then split integral into part near $t = 0$ and part away from $t = 0$, integrate by parts in the $t \geq \varepsilon$ part of the integral.

4.4. **Mean value property.** The heat equation does have a "parabolic" mean value property. It is not often used and not easy to generalize so we will not do it in lecture, you can find it in Evans.

4.5. **Maximum principle and uniqueness.** Heat equation initial - boundary value problem (IBVP) (Dirichlet data)

$$\begin{cases} \partial_t u - \Delta u = f(x) & \text{in } U \times (0,\infty) \\ u(x,0) = g(x) & \text{in } U \\ u(x,t) = h(x,t) & \text{on } \partial U \times (0,\infty). \end{cases}$$

Natural space-time domains are of the form $U_T = U \times (0,T]$ called *parabolic cylinders*. The *parabolic boundary* is defined

$$\partial_p U_T = \Gamma_T = \overline{U}_T \setminus U_T = \overline{U} \times \{t = 0\} \cup \partial U \times [0,T].$$

IBVP's naturally assign boundary data on the parabolic boundary.

In considering uniqueness, as usual we can use linearity to reduce to considering the case of zero boundary data.

Say that $u \in C^{2,1}(U_T)$ is a **subsolution** of the heat equation in a parabolic domain U_T if

$$\partial_t u - \Delta u \leq 0$$

(strict if inequality is strict). **Supersolution** is defined symmetrically. Note we don't have a weak formulation of this right now (it is possible but more technically difficult than for Laplace).

Lemma 3. *If u and v are respectively a subsolution and a supersolution in a bounded parabolic domain $U_T = U \times (0,T]$ with $u \leq v$ on $\partial_p U_T$ then $u \leq v$ in U_T.*

Proof. 1. First consider the case when u is a strict subsolution and $u < v$ for $(x,t) \in \partial_p U_T$. Then consider the first time that u crosses v from below

$$t_* = \inf\{.t > 0 : \inf(v - u) \leq 0\}$$

by continuity there is a point $x_* \in \overline{U}$ with

$$u(x,t_*) \leq v(x,t_*) \quad \text{for} \quad x \in U \quad \text{and} \quad u(x_*,t_*) = v(x_*,t_*).$$

Note that $x_* \in U$ because $u < v$ on $\partial U \times [0,T]$. Then $w(x,t_*) = (v-u)(x,t_*)$ has a spatial minimum at x_* so

$$\Delta w(x_*,t_*) \geq 0.$$

Also $w > 0$ for $t < t_*$ so

$$\partial_t w(x_*,t_*) = \lim_{h \to 0} \frac{w(x_*,t_*) - w(x_*,t_* - h)}{h} \leq 0.$$

Thus

$$0 \geq [\partial_t w - \Delta w](x_*, t_*) = [\partial_t v - \Delta v](x_*, t_*) - [\partial_t u - \Delta u](x_*, t_*) > 0$$

,using the supersolution / strict subsolution properties of v and u respectively, which is a contradiction.

2. Now consider the case where the inequalities are not strict. In this case we can take

$$u_\varepsilon(x,t) = u(x,t) - \varepsilon - \varepsilon t$$

which has $u_\varepsilon < u \leq v$ on $\partial_p U_T$ and

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = \partial_t u - \varepsilon - \Delta u \leq -\varepsilon < 0$$

so u_ε is a strict subsolution which is strictly below u on $\partial_p U_T$ and we can apply the first part of the proof to find $u_\varepsilon \leq v$ in U_T and then send $\varepsilon \to 0$ to get the result. □

4.6. **Maximum principle in $\mathbb{R}^n$.** Applying the previous argument in the whole space is more difficult, we need some information to control the growth of u at ∞ otherwise maximum principle can fail. For example recall that when we derived the fundamental solution we came up with a solution $u(x,t) = t^{-n/2} w(|x|/t^{1/2})$, now we look for a solution of the form $u(x,t) = (-t)^{-n/2} w(|x|/(-t)^{1/2})$ defined for $t < 0$ and going through the derivative one finds that

$$w' = \frac{1}{2} r w$$

resulting in a solution

$$u(x,t) = \frac{1}{(-t)^{n/2}} e^{\frac{|x|^2}{4(-t)}} \quad \text{for} \quad t < 0.$$

This solution is "non-physical" in the sense that it starts out small near the origin at (say) time $t = -1$ and then grows creating a singularity $t = 0$. This is caused by the large growth at ∞, energy is being pumped in from ∞ to create a singularity.

Actually the existence of such a *positive* singular solution allows us to rule out non-uniqueness for solutions which grow more slowly at ∞ than this special solution.

Other examples of this general principle (exercises):

- For Laplace equation $\Delta u = 0$ in half space $\{x_n > 0\}$ with zero Dirichlet data $u(x', 0) = 0$ can use the harmonic function $\phi(x) = x_n$ to rule out sub-linearly growing solutions.
- For Laplace equation $\Delta u = 0$ in strip domain $\{0 < x_n < 1\}$ with zero Dirichlet data $u(x', 0) = u(x', 1) = 0$ can use the harmonic function $\phi(x) = \frac{1}{2(n-1)} |x'|^2 - \frac{1}{2}(x_n - \frac{1}{2})^2 + \frac{1}{8}$ to rule out sub-quadratically growing solutions.

Lemma 4. *Suppose that $u \in C^{2,1}(\mathbb{R}^n \times [0,T])$ solves*

$$u_t - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T) \quad \text{and} \quad u(x,0) = g$$

and satisfies the growth estimate

$$u(x,t) \leq Ae^{a|x|^2}$$

for some constants $A, a > 0$. Then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup g.$$

Suffices to consider the case $u(x,0) \leq 0$ by considering $u(x,t) - \sup g$.

Proof. Assume $4aT < 1$ so that $4a(T+\varepsilon) = 1-\gamma$ for some fixed small $\varepsilon > 0$ and some $\gamma > 0$. Now the function

$$v(x,t) = \frac{\delta}{(T+\varepsilon-t)^{n/2}} e^{\frac{|x|^2}{4(T+\varepsilon-t)}}$$

is a solution of the heat equation on $[0,T]$ with $v(x,0) \geq 0 \geq u(x,0)$. On the other hand for any $R > 0$ and $x \in \partial B(0,R)$

$$\begin{aligned} u(x,t) &\leq Ae^{a|x|^2} = Ae^{aR^2} \\ &= Ae^{(1-\gamma)\frac{R^2}{4(T+\varepsilon)}} \\ &\leq Ae^{(1-\gamma)\frac{R^2}{4(T+\varepsilon-t)}} \\ &= Ae^{-\gamma\frac{R^2}{4(T+\varepsilon-t)}} e^{\frac{|x|^2}{4(T+\varepsilon-t)}} \\ &\leq \frac{A}{\delta}(T+\varepsilon)^{n/2} e^{-\gamma\frac{R^2}{4(T+\varepsilon-t)}} v(x,t) \leq \mu v(x,t) \end{aligned}$$

for $\mu \in (0,1)$ and $R \geq R_0(\mu) > 1$ sufficiently large depending on μ. Thus by comparison principle in $B(0,R) \times (0,T]$ (bounded domain) for each $R \geq R_0(\mu)$ we have $u(x,t) \leq \mu v(x,t)$ on $\mathbb{R}^n \times (0,T]$. Since $\mu > 0$ was arbitrary $u(x,t) \leq 0$ on $\mathbb{R}^n \times (0,T]$.

Finally if $4aT > 1$ we simply apply the argument repeatedly on intervals of length $T_1 = \frac{1}{8a}$. □

4.7. **Energy methods.** One thing we can do is uniqueness of IBVP by energy argument

$$\begin{cases} \partial_t u - \Delta u = f(x) & \text{in } U \times (0,T] \\ u(x,0) = g(x) & \text{in } U \\ u(x,t) = h(x,t) & \text{on } \partial U \times (0,T]. \end{cases}$$

If we have two solutions u and v then $w = u - v$ solves

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } U \times (0,T] \\ w(x,0) = 0 & \text{in } U \\ w(x,t) = 0 & \text{on } \partial U \times (0,T]. \end{cases}$$

Compute time derivative of the L^2 norm $e(t) = \int_U u^2 \, dx$

$$\frac{d}{dt}\int_U u^2 \, dx = \int_U 2u\partial_t u \, dx = \int_U 2u\Delta u \, dx = \int_{\partial U} 2u\frac{\partial u}{\partial \nu} - 2\int_U |Du|^2 \, dx = -2\int_U |Du|^2 \, dx.$$

Since the L^2 norm at $t = 0$ is zero, it is also zero at all positive times (it is non-negative and non-positive).

Could also compute time derivative of Dirichlet energy

$$\frac{d}{dt}\int_U |Du|^2 \, dx = \int_U 2Du\cdot D\partial_t u \, dx = \int_{\partial U} 2\partial_t u\frac{\partial u}{\partial \nu} dS - 2\int_U u_t \Delta u \, dx = -2\int_U |\Delta u|^2 \, dx$$

where we used that $u(x,t) = 0$ for $x \in \partial U$ and all $t \in (0,T]$ so $\partial_t u(x,t) = 0$ as well on $\partial U \times (0,T]$.

Theorem 5 (Backwards uniqueness)**.** *Suppose that u in $C^2(\overline{U}_T)$ solves the heat equation with $u = 0$ on $\partial U \times [0,T]$ and*

$$u(x,T) = 0$$

then $u \equiv 0$ in U_T.

Proof. Define $e(t) = \int u^2 \, dx$ as before we saw

$$\dot{e}(t) = -2\int_U |Du|^2 \, dx$$

and

$$\ddot{e}(t) = 4\int_U |\Delta u|^2 \, dx.$$

On the other hand

$$\int_U |Du|^2 \, dx = -\int_U u\Delta u \, dx \leq (\int_U u^2 \, dx)^{1/2}(\int_U (\Delta u)^2 \, dx)^{1/2}$$

so

$$(\dot{e}(t))^2 \leq e(t)\ddot{e}(t)$$

This is some kind of convexity statement.

If $e(t) \equiv 0$ we are done, otherwise there is an interval $[t_1, t_2]$ with $e(t) > 0$ on $[t_1, t_2)$ and $e(t_2) = 0$. Note

$$f(t) = \log e(t)$$

has

$$\ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \geq 0$$

so f is convex on $(t_1, t_2]$ and $e(t)$ is log convex and for any $t_1 < t < t_2$

$$f(\lambda t_1 + (1-\lambda)t) \leq \lambda f(t_1) + (1-\lambda)f(t)$$

or

$$0 \leq e(\lambda t_1 + (1-\lambda)t) \leq e(t_1)^\lambda e(t)^{1-\lambda}$$

but now $e(t)$ is bounded and continuous so we can send $t \nearrow t_2$ to find

$$0 \leq e(\lambda t_1 + (1-\lambda)t_2) \leq e(t_1)^\lambda e(t_2)^{1-\lambda} = 0$$

$\square$

4.8. **Connection with random walks/Brownian motion.** Let's recall our setting from before of a simple random walker $X_t(x)$ on $\mathbb{Z}^n$ starting from a point x. Each step $X_{t+1}(x) - X_t(x) = \xi_t$ is independent of all other steps and is distributed uniformly on the $2n$ neighbors of 0 which are $\pm e_i$ for $1 \leq i \leq n$. Now instead of considering the exit probabilities we will just look at the distribution

$$K_t(x,y) = \mathbb{P}(X_t(x) = y) \ \text{ for } x, y \in \mathbb{Z}^n.$$

Let's check that $K_t(x,y)$ solves a discrete heat equation in both variables,

$$K_{t+1}(x,y) = \mathbb{P}(X_{t+1}(x) = y) = \sum_{z \sim y} \mathbb{P}(X_t(x) = z)\mathbb{P}(\xi_t = (y-z)) = \frac{1}{2d}\sum_{z\sim y} K_t(x,z)$$

subtracting $K_t(x,y)$ from both sides we get

$$K_{t+1}(x,y) - K_t(x,y) = \frac{1}{2d}\sum_{z\sim y}(K_t(x,z) - K_t(x,y)) = \Delta_y K_t(x,y)$$

we could have also computed

$$K_{t+1}(x,y) = \mathbb{P}(X_{t+1}(x) = y) = \sum_{z\sim x} \mathbb{P}(X_t(z) = y)\mathbb{P}(\xi_1 = (z-x)) = \Delta_x K_t(x,y) + K_t(x,y)$$

to get the equation in the other variable.

The function $K_t(x,y)$ is the discrete analogue of the heat kernel.

A general initial value problem for the heat equation corresponds to a "payoff" problem at time zero (draw picture)

$$u(t,x) = \mathbb{E}g(X_t(x))$$

which can be represented in terms of the heat kernel

$$u(t,x) = \sum_y g(y)\mathbb{P}(X_t(x) = y) = \sum_y K_t(x,y)g(y).$$

This is the discrete analogue of the heat kernel formula we saw before.

In the continuum case we study Brownian motion instead and

$$K_t(x,E) = \mathbb{P}(B_t(x) \in E) = \int_E \frac{1}{(2\pi t)^{n/2}} e^{-|x-y|^2/2t} dy$$

which you will recognize as the heat kernel (up to a change of variance).

Initial value problems for the heat equation can be interpreted then as

$$u(t,x) = \mathbb{E}g(B_t(x)) = \int_{\mathbb{R}^n} g(y)K_t(x,y)\ dy$$

which we know solves the heat equation $u_t = \frac{1}{2}\Delta u$.

4.9. **Regularity.** See Evans for proofs. Statement: if $u \in C^{2,1}(U_T)$ solves the heat equation then $u \in C^\infty(U_T)$. There are analogues of elliptic estimates as well that show that scaling of derivative estimates in terms of domain size / number of space/time derivatives.

5. Wave equation

General ideas: time reversible, energy conserving, finite speed of propagation.

Homogeneous wave equation is

$$u_{tt} - \Delta u = 0$$

sometimes the wave operator is called the D'Alembertian and written

$$\Box u = u_{tt} - \Delta u$$

Thinking of $u_{tt} = \Delta u$ as a second order linear ODE on function space we expect that a natural initial data problem for the wave equation will involve assigning data for both $u(x,0)$ and $u_t(x,0)$.

5.1. **Energy conservation (whole domain).** Energy is kinetic plus potential

$$E(t) = \frac{1}{2}\int_U u_t^2 + |Du|^2 \ dx.$$

Show that $E'(t) = 0$ with Dirichlet/Neumann data.

5.2. **d'Alembert's solution in $n = 1$.** PDE can be "factored"

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = u_{tt} - c^2 u_{xx} = 0.$$

This equation is much simpler in the characteristic coordinates $\xi = x + ct$ and $\eta = x - ct$ since this says

$$\partial^2_{\xi\eta} u = 0$$

i.e.

$$u(x,t) = F(\xi) + G(\eta) = F(x - ct) + G(x + ct).$$

Note that this is a linear combination / superposition of a left moving travelling wave and a right move travelling wave.

We just need to solve for F and G given some initial data problem

$$u(x,0) = f(x) \ \text{ and } \ u_t(x,0) = g(x).$$

Then

$$F(x) + G(x) = f(x) \ \text{ and } \ cG'(x) - cF'(x) = g(x).$$

This leads to the equations

$$F(x) + G(x) = f(x) \ \text{ and } \ G(x) - F(x) = \frac{1}{c}\int_0^x g(y) \ dy + A$$

so

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(y) \ dy - \frac{1}{2}A$$

and

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(y) \ dy + \frac{1}{2}A$$

and

$$u(x,t) = F(x-ct)+G(x+ct) = \frac{1}{2}f(x-ct)+\frac{1}{2}f(x+ct)+\frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\ dy.$$

Example: triangle initial data.

5.3. **Non-homogeneous problem $n = 1$.** Duhamel's formula

$$u_{tt} - u_{xx} = h(x,t)$$

with zero initial data solution. Think of this as a linear ODE

$$\frac{d}{dt}\begin{bmatrix} u \\ u_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{bmatrix}\begin{bmatrix} u \\ u_t \end{bmatrix} + \begin{bmatrix} 0 \\ f(x,t) \end{bmatrix}$$

with operator

$$L = \begin{bmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{bmatrix}.$$

By Duhamel (first component of solution) is

$$u(x,t) = \int_0^t \left(e^{Lt}\begin{bmatrix} 0 \\ f(\cdot,s) \end{bmatrix}\right)_1\ ds = \int_0^t u(x,t;s)\ ds$$

where $u(x,t;s)$ solves

$$u_{tt} - u_{xx} = 0 \ \text{ with } \ u(x,0) = 0 \ \text{ and } \ u_t(x,0) = f(x,s).$$

i.e.

$$u(x,t) = \int_0^t u(x,t;s)\ ds = \int_0^t \int_{x-cs}^{x+cs} \frac{1}{2c}f(y,s)\ dy\ ds$$

5.4. **Domain of dependence / region of influence.** Draw backwards light cone from (x_0, t_0).

Note that initial data problem $u(x_0, t_0)$ depends only on values of initial data inside (x_0-ct, x_0+ct). Note that inhomogeneous problem only depends on the inhomogeneity inside the backwards light cone.

State a theorem to this effect (two different data that agree in backwards light cone from (x_0, t_0) produce same solution in that backwards light cone).

State and prove the same theorem in all $n \geq 1$ using the energy method computing time derivative of energy inside the cone

$$E(t) = \int_{|x-x_0|<ct} \frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2\ dx.$$

Conclude uniqueness.

Side note: Moving domain differentiation formula

$$\frac{d}{d\tau}\int_{U(\tau)} f(x,\tau)\ dx = \int_{\partial U(\tau)} f\mathbf{V}\cdot n dS + \int_{U(\tau)} \partial_\tau f(x,\tau)\ dx.$$

Proof: Assume we can write

$$U(\tau) = \Phi_\tau(U)$$

where Φ_τ is the flow map for the ODE

$$\dot{\Phi}_\tau = \mathbf{V}(\Phi_\tau)$$

and by Liouville formula

$$\frac{d}{dt}|\det(\Phi_\tau(x))| = (\nabla \cdot V)(\Phi_\tau(x))|\det(\Phi_\tau(x))|$$

Then

$$\int_{U(\tau)} f(y)\ dy = \int_U f(\Phi_\tau^{-1}(y))|\det(\Phi_\tau(x))|dx$$

and

$$\begin{aligned}\frac{d}{d\tau}\int_{U(\tau)} f(y)\ dy &= \int_U \nabla f(\Phi_\tau(x)) \cdot \frac{d}{d\tau}\Phi_\tau(x)|\det(\Phi_\tau(x))|dx \\ &\quad \cdots + \int_U f(\Phi_\tau(x))(\nabla \cdot V)(\Phi_\tau(x))|\det(\Phi_\tau(x))|dx \\ &= \int_U \nabla f(\Phi_\tau(x)) \cdot (\frac{d}{d\tau}\Phi_\tau(x) - V(\Phi_\tau(x)))|\det(\Phi_\tau(x))|dx \\ &\quad \cdots + \int_{\partial U} f(\Phi_\tau(x))V(\Phi_\tau(x)) \cdot n|\det(\Phi_\tau(x))|dS(x)\end{aligned}$$

5.5. **Reflection method in $n = 1$.** Boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = 0 & \text{on } \{x = 0\} \times (0,\infty). \end{cases}$$

Extend u, g, and h by odd reflection across $x = 0$ to $\tilde{u}$, $\tilde{g}$ and $\tilde{h}$. Get a solution of wave equation on whole line and apply d'Alembert

$$u(x,t) = \begin{cases} \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\ dy & x \geq t > 0 \\ \frac{1}{2}[g(x+t) - g(t-x)] + \frac{1}{2}\int_{t-x}^{x+t} h(y)\ dy & 0 \leq x \leq t. \end{cases}$$

Intuition: initial data creates left and right moving waves, the left moving wave reflects off of the boundary at $x = 0$ and moves right. Draw spacetime picture.

5.6. **Method of spherical means.** Solution of the wave equation u in $\mathbb{R}^n$ define

$$U(x;t,r) = ⨍_{\partial B(x,r)} u(y,t)dS(y).$$

Similar definitions for initial data

$$G(x;r) \text{ and } H(x;r).$$

For a fixed x the spherical means U solve

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ U = G,\ U_t = H & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ U_r = 0 & \text{on } \{x = 0\} \times (0,\infty). \end{cases}$$

PDE is called Euler-Poisson-Darboux equation.

Proof:

Same computation as in proof of MVT for harmonic functions we found

$$U_r = \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)\ dy.$$

Second derivative

$$U_{rr} = \int_{\partial B(x,r)} \Delta u dS + (\frac{1}{n} - 1) \fint_{B(x,r)} \Delta u(y,t)\ dy$$

etc can compute higher derivatives and show that limit exists at $r = 0$.

From above

$$U_r = \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)\ dy = \frac{r}{n} \fint_{B(x,r)} u_{tt}\ dy = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} u_{tt}(y,t)\ dy$$

so

$$r^{n-1} U_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt}(y,t)\ dy$$

and

$$(r^{n-1} U_r)_r = \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}(y,t)\ dy = r^{n-1} U_{tt}.$$

5.7. **Kirchoff's formula.** We will find a distinction between even and odd dimensions, higher dimensions are more difficult so we will stick to $n = 2$ and $n = 3$ (physical dimensions). Start with $n = 3$ and define

$$\tilde{U} = rU, \quad \text{etc.}$$

We claim

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ \tilde{U} = \tilde{G},\ \tilde{U}_t = \tilde{H} & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ \tilde{U} = 0 & \text{on } \{x = 0\} \times (0,\infty). \end{cases}$$

which leads to the formula for $0 \le r \le t$

$$\tilde{U}(x; r, t) = \frac{1}{2}[\tilde{G}(x; r+t) - \tilde{G}(x; t-r) + \frac{1}{2} \int_{t-r}^{t+r} \tilde{H}(x; y)\ dy.$$

Then

$$u(x,t) = \lim_{r \to 0} \frac{\tilde{U}(x; r, t)}{r} = \lim_{r \to 0} \left[\frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{2r} + \frac{1}{2r} \int_{t-r}^{t+r} \tilde{H}(y)\ dy \right] = \tilde{G}'(x;t) + \tilde{H}(x;t).$$

This gives the equation

$$u(x,t) = \frac{\partial}{\partial t} \left(\fint_{\partial B(x,t)} t g(y) dS(y) \right) + t \fint_{\partial B(x,t)} h(y)\ dS(y).$$

further computation gives

$$u(x,t) = \fint_{\partial B(x,t)} t h(y) + g(y) + Dg(y)(y - x) dS(y)$$

which is called **Kirchhoff's formula**.

Make note of the domain of dependence of $u(x,t)$!

5.8. **Method of descent.** There is not a clean transformation from EPD to wave equation in $n=2$, however we can still use the $3-d$ formula! Given initial data in $n=2$ extend it to be defined in an "imaginary" third variable with derivative zero in that direction. Precisely for $f \in \{u,g,h\}$ define

$$\bar{f}(x_1,x_2,x_3,t) = f(x_1,x_2,t).$$

Then $\bar{u}$ solves the wave equation in $n=3$ with initial data $(\bar{g},\bar{h})$.

Write

$$\fint_{\partial \bar{B}(\bar{x},t)} \bar{g} d\bar{S} = \frac{1}{4\pi t^2}\int_{\partial \bar{B}(\bar{x},\bar{t})} \bar{g} d\bar{S} = \frac{1}{4\pi t^2}\int_{B(x,t)} g(y)(1+|D\gamma(y)|^2)^{1/2} d\bar{S}$$

where $\gamma(y) = (t^2-|y-x|^2)^{1/2}$. Note

$$(1+|D\gamma(y)|^2)^{1/2} = (1+|x-y|^2/(t^2-|x-y|^2)^{1/2})^{1/2} = t(t^2-|x-y|^2)^{1/2}.$$

Finish computation.

End result

$$u(x,t) = \frac{1}{2}\int_{B(x,t)} \frac{tg(y)+t^2h(y)+tDg(y)\cdot(y-x)}{(t^2-|y-x|^2}^{1/2} dy.$$

Conclude by discussing **Huygen's principle**.

Also point out in Kirchhoff's formula the appearance of Dg, solution may not be as regular as the initial data (information about $D^k u$ will depend on $D^{k+1}g$).

5.9. **Frequency, group velocity, phase velocity, dispersion relations.** Given a PDE involving (x,t) space-time variables we refer to a solution u as a **travelling wave** solution with **speed** c and **profile** v if

$$u(x,t) = v(x-ct).$$

In higher dimensions a solution of the form

$$u(x,t) = v(p\cdot x - ct)$$

is called a plane wave or travelling wave travelling in the direction $p/|p|$ with **wave speed** $c/|p|$.

We have seen solutions of this form already in the transport and wave equations, and we will see more. A particular form of travelling wave which is quite important (given connection with the Fourier transform) is the **exponential solution**

$$u(x,t) = e^{i(k\cdot x-\omega t)}.$$

Here $k \in \mathbb{R}^d$ is called the **wave number** or **wave vector** (in $n \geq 2$), ω is the **time frequency**. The relationship

$$k \mapsto \omega(k)$$

is called the **dispersion relation** and will be enforced by the equation. The **phase velocity** is

$$v_p(k) = \frac{\omega(k)}{|k|}$$

is the speed of propagation in the direction $k/|k|$. If the dispersion relation is nonlinear the phase velocity will depend nontrivially on the wave number. In this case the equation is called **dispersive**.

For example in the wave equation

$$u_{tt} - c^2 \Delta u = (-\omega^2 + c^2|k|^2)u = 0$$

so to solve the wave equation we must have

$$\omega = \pm c|k|.$$

The wave speed is $|\omega|/|k| = c$ no matter the choice of wave number/frequency so the wave equation has a trivial dispersion relation.

For the Klein-Gordon equation (equation for wave function of relativistic free quantum particle with mass m)

$$u_{tt} - \Delta u + m^2 u = 0$$

we plug in the same ansatz

$$0 = u_{tt} - \Delta u + m^2 u = (-\omega^2 + |k|^2 + m^2)u = 0$$

so the dispersion relation is

$$\omega = \pm\sqrt{|k|^2 + m^2}$$

which is **nonlinear** and the equation is called **dispersive**. Here small wave number $|k|$ i.e. slower spatial oscillation leads to a faster propagation speed

$$v_p(k) = \sqrt{1 + \frac{m^2}{|k|^2}}$$

In the **Schrödinger equation** from quantum mechanics (non-relativistic free particle)

$$iu_t - \Delta u = 0$$

we find the dispersion relation

$$\omega = |k|^2$$

and

$$v_p(k) = |k|$$

so the equation is **dispersive** and higher wave numbers propagate *faster*.

5.10. **Wave packets.** Of course the exponential form of a wave solution is very specific, in greater generality we prefer to look at **wave packets** or **wave groups** one way to represent this idea is by linear combination of pure phase waves

$$u(x,t) = \int_{\mathbb{R}^n} e^{i(k\cdot x-\omega(k)t)}a(k)\ dk.$$

This is a linear combination of solutions so, as long as our equation was linear, it will solve as well. (Note: this is basically the same as solving the PDE in general by Fourier transform which we will look at later).

To understand the propagation speed of this wave packet we look along a space-time ray $x = vt$ and send $t \to \infty$:

$$\int_{\mathbb{R}^n} e^{i(k\cdot x-\omega(k)t)}a(k)\ dk = \int_{\mathbb{R}^n} e^{it(k\cdot v-\omega(k))}a(k)\ dk = I(t;v).$$

The integrand is highly oscillatory as $t \to \infty$. It turns out that the dominant term in the asymptotic expansion for $I(t;v)$, the **principal of stationary phase**, comes from wave numbers k for which $0 = D_k(k \cdot v - \omega(k)) = v - D_k\omega(k)$ which leads to the definition of the **group velocity**

$$v_g(k) = D\omega(k).$$

(i.e. the wave numbers which are seen along the ray $x = vt$ are those with $D\omega(k) = v$ so we say that those wave numbers propagate with group velocity v).

5.11. **Quick primer on (non)-stationary phase.** Oscillatory integral

$$I(t) = \int_{\mathbb{R}^n} e^{it\phi(x)}a(x)\ dx.$$

We will consider the cases $\phi(x) = p \cdot x$ linear and $\phi(x) = \frac{1}{2}x^TAx$ quadratic (model cases for phase functions ϕ which have $D\phi \neq 0$ and a single **stationary point** $D\phi(0) = 0$). We will also just do $1-d$ case.

First case is **non-stationary phase** assume a is smooth and compactly supported

$$I(t) = \int_{\mathbb{R}} e^{itpx}a(x)\ dx \quad \text{with} \quad p \neq 0.$$

In this case we integrate by parts putting derivatives onto a

$$I(t) = \int_{\mathbb{R}} \left(\frac{1}{itp}\right)^m e^{itpx}\left(\frac{\partial}{\partial x}\right)^m a(x)\ dx$$

so

$$|I(t)| \leq Ct^{-m}.$$

Next case is existence of a stationary point of the phase function model scenario

$$I(t) = \int_{\mathbb{R}} e^{it\frac{1}{2}\alpha x^2}a(x)\ dx \quad \text{with} \quad \alpha \neq 0$$

with a smooth and supported in $[-1,1]$.

6. Fourier methods

Fourier transform on $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ taking complex values

$$\hat{f}(\xi) = \mathcal{F}(f) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(x)\ dx$$

inverse Fourier transform

$$\check{f}(x) = \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot x} f(\xi)\ d\xi.$$

Intuitively can think of $e^{2\pi i \xi \cdot x}$ as an orthonormal basis of $L^2(\mathbb{R}^n)$ over $\xi \in \mathbb{R}^n$, FT computes the representation in the Fourier basis. This is not quite sensible because the basis functions are not in L^2. This is, however, true when the domain is $\mathbb{T}^n$ instead of $\mathbb{R}^n$ (basis consists of complex exponentials with $\xi \in \mathbb{Z}^n$).

(Plancherel)

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}\ dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi$$

(Derivatives) Fourier transform diagonalizes differential operators

$$\widehat{D^\alpha f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} D^\alpha f(x)\ dx = (2\pi i \xi)^\alpha \hat{f}(\xi)$$

(Convolution)

$$\widehat{(f \star g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

(Translations / modulations)

$$\widehat{f(x+y)} = e^{2\pi i \xi \cdot y} \hat{f}(\xi) \quad \text{and} \quad \mathcal{F}(e^{2\pi i \eta \cdot x} f(x)) = \hat{f}(\xi - \eta).$$

(Delta mass / pure mode)

$$\mathcal{F}(e^{2\pi i \eta \cdot x}) = \delta_\eta(\xi)$$

Fourier transform is a powerful tool especially for linear constant coefficient PDE.

6.1. **Fundamental solution of heat equation.** Say we are looking to solve

$$u_t - \Delta u = 0 \ \text{ for } t > 0 \text{ with } \ u(x,0) = g(x).$$

Let's take the Fourier Transform of this equation in the spatial variables (only!)

$$\hat{u}_t - (2\pi i \xi) \cdot (2\pi i \xi) \hat{u} = \hat{u}_t + 4\pi^2 |\xi|^2 \hat{u} = 0$$

and the initial data transforms

$$\hat{u}(\xi, 0) = \hat{g}(\xi).$$

This is an ordinary differential equation for each ξ!

$$\hat{u}(\xi, t) = e^{-4\pi^2 |\xi|^2 t} \hat{g}(\xi).$$

Then

$$u = \mathcal{F}^{-1} \hat{u} = \mathcal{F}^{-1}(e^{-4\pi^2 |\xi|^2 t}) \star g.$$

Thus we need to compute the inverse Fourier transform of this Gaussian which we can do one variable at a time

$$\mathcal{F}^{-1}(e^{-4\pi^2|\xi|^2 t}) = \int_{\mathbb{R}} e^{2\pi i\xi x - 4\pi^2|\xi|^2 t} d\xi = \int_{\mathbb{R}} e^{2\pi i\xi x - 4\pi^2|\xi|^2 t} d\xi$$

completing the square in the exponential

$$2\pi i\xi x - 4\pi^2|\xi|^2 t = (2\pi i\xi t^{1/2} + \frac{x}{2t^{1/2}})^2 - \frac{|x|^2}{4t}$$

so

$$\int_{\mathbb{R}} e^{2\pi i\xi x - 4\pi^2|\xi|^2 t} d\xi = e^{-|x|^2/4t} \int_{\mathbb{R}} e^{(2\pi i\xi t^{1/2} - \frac{x}{2t^{1/2}})^2} d\xi$$

and with the remaining integral

$$\int_{\mathbb{R}^n} e^{(2\pi i\xi t^{1/2} - \frac{x}{2t^{1/2}})^2} d\xi = \frac{1}{2\pi t^{1/2}} \int_{\mathbb{R}} e^{(iy - \frac{x}{2t^{1/2}})^2} dy = \frac{1}{2\pi t^{1/2}} \int_{\mathbb{R}} e^{-y^2} dy$$

(by moving the contour).

6.2. **Schrödinger equation.** Initial data $g \in L^2(\mathbb{R}^n)$ complex valued

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \end{cases}$$

Solution u is interpreted as quantum mechanical *wave function.* The magnitude squared $|u|^2$ is the probability density function for the location of the particle.

Solution formula

$$u(x,t) = \frac{1}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} e^{i|x-y|^2/4t} g(y)\ dy$$

can find this by plugging in *it* into heat equation fundamental solution turns out to work and solve Schrödinger equation. Expand the square in the exponential and apply Plancherel to see that Schrödinger operator preserves L^2 norm. Can also do direct computation.

Notes: Schrödinger equation is time reversible.

6.3. **More on wave packets.** Consider a *modulated* plane wave initial data

$$u_0(x) = \varphi(x)e^{2\pi i k\cdot x}$$

where φ could be (for example) a Gaussian $\varphi(x) = e^{-|x|^2/2}$ so that $\hat{\varphi}$ is also a Gaussian with variance ~ 1. Actually it will be simpler to assume that $\hat{\varphi}$ is smooth and supported in a unit neighborhood of the origin in Fourier space which we can guarantee by choosing $\hat{\varphi}$ first and then taking the inverse transform.

We are solving some linear PDE with dispersion relation

$$k \mapsto \omega(k)$$

i.e. $e^{2\pi i(k\cdot x - \omega(k)t)}$ solves the PDE for each wave number k (added in the factors of 2π to match our Fourier transform.

Then

$$u_0(x) = \mathcal{F}^{-1}(\hat{u}_0) = \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot x} \hat{\varphi}(\xi - k) d\xi$$

and by linearity the solution of our PDE is

$$u(x,t) = \int_{\mathbb{R}^n} e^{2\pi i (\xi \cdot x - \omega(\xi) t)} \hat{\varphi}(\xi - k) d\xi.$$

Let

$$V = D\omega(\{\xi : |\xi - k| \leq 1\}).$$

Then for any $v \notin V$

$$u(x, vt) = \int_{\mathbb{R}^n} e^{2\pi i t (\xi \cdot v - \omega(\xi))} \hat{\varphi}(\xi - k) d\xi$$

the phase function $\xi \cdot v - \omega(\xi)$ is non-stationary on the support of $\hat{\varphi}$ so $u(x, vt) = O(t^{-m})$ for any $m \geq 1$ as $t \to \infty$. The non-trivial propagation speeds of the wave packet are then limited to V.

7. Nonlinear first order equations

. We start with a general form of nonlinear equation

$$F(Du, u, x) = 0 \ \text{ in } \ U \subset \mathbb{R}^n \ \text{ with } \ u = g \ \text{ on } \ \Gamma \subset \partial U.$$

We start with general techniques, the method of characteristics, which gives a classical solution but the solution may develop singularities outside of a neighborhood of the boundary. Then will move to special types of equations where we can use "physical principles" to define a notion of weak solution which is global in time (Hamilton-Jacobi equations arising from dynamics / control theory, Conservation laws arising from continuum mechanics).

We write $F(p, z, x)$ and assume F is smooth with $D_p F = (F_{p_1}, \ldots, F_{p_n})$, $D_z F = F_z$, and $D_x F = (F_{x_1}, \ldots, F_{x_n})$.

Note one of the x variables could be a "time".

7.1. Method of characteristics.

. The idea is to find curves $x(s)$ linking interior points x to boundary points $x_0 \in \Gamma$ along which we can compute u (by a system of ODE).

Let $x(s)$ be a parametrized curve $s \in I$ some parameter interval. Define

$$z(s) = u(x(s)) \ \text{ and } \ p(s) = Du(x(s)).$$

We need to choose the curve in such a way that the ODE for (x, z, p) close.

The ODE for $\dot{x}(s) =?$ is to be determined, given that we would find

$$\dot{z}(s) = Du(x(s)) \cdot \dot{x}(s) = p(s) \cdot \dot{x}(s) \ \text{ and } \ \dot{p}_i(s) = D^2_{ij} u(x(s)) \dot{x}_j(s).$$

The latter equation is a bit worrisome because it involves second order derivatives, but we can use the PDE to make some identity by differentiating the PDE with respect to x_i

$$\begin{aligned}0 &= (F(Du,u,x))_{x_i}\\ &= \sum_j D_{p_j}F(Du,u,x)D^2_{ij}u + D_zF(Du,u,x)D_iu + D_{x_i}F(Du,u,x)\end{aligned}$$

this allows us to eliminate second order terms the equation for $\dot{p}$ if we set

$$\dot{x}(s) = D_pF(p(s),z(s),x(s))$$

which reduces the p equation to

$$\dot{p}(s) = -D_xF(p(s),z(s),x(s)) - D_zF(p(s),z(s),x(s))p(s)$$

summarizing we get the **characteristic equations**

$$\begin{cases}\dot{x}(s) = D_pF(p(s),z(s),x(s))\\ \dot{z}(s) = p(s)\cdot D_pF(p(s),z(s),x(s))\\ \dot{p}(s) = -D_xF(p(s),z(s),x(s)) - D_zF(p(s),z(s),x(s))p(s)\end{cases}$$

and for any solutions of this ODE system it is guaranteed that

$$F(p(s),z(s),x(s)) = 0 \ \text{ for } \ x \in I.$$

The issue now is how to use these equations to solve PDE BVP.

7.2. **Examples.** (Linear equation, Transport/growth/decay equation)

$$F(Du,u,x) = b(x)\cdot Du + c(x)u = 0$$

Then

$$F(p,z,x) = b(x)\cdot p + c(x)z$$

so

$$F_p = b(x) \ \text{ and } \ F_z = c(x)$$

the equation for p will not be necessary for linear/semi-linear/quasi-linear problems

$$\dot{x} = b(x), \ \text{ and } \dot{z} = p(s)\cdot b(x) = -c(x)z.$$

Note: one variable could be a time $(x_1,\ldots,x_n,t)$ then usually $b_{n+1}(x) \equiv 1$ and the associated characteristics is $\dot{t} = 1$ identifying the parameter s with the t variable (up to a translation).

Solve example

$$\begin{cases}x_1u_{x_2} - x_2u_{x_1} = u & \text{in } U = \{x_1 > 0, x_2 > 0\}\\ u = g & \text{on } \Gamma = \{x_1 > 0, x_2 = 0\} \subset \partial U.\end{cases}$$

(Quasi-linear equation) general quasi-linear first order equation has the form

$$F(Du,u,x) = b(x,u(x))\cdot Du(x) + c(x,u(x)) = 0.$$

Again the equations close without the p equation

$$F_p = b(x,z) \ \text{ and } \ F_z = c(x,z)$$

so

$$\dot{x} = b(x,z), \quad \text{and } \dot{z} = p(s) \cdot b(x,z) = -c(x,z).$$

Example:

$$\begin{cases} u_{x_1} + u_{x_2} = u^2 & \text{in } U = \{x_2 > 0\} \\ u = g & \text{on } \Gamma = \{x_2 = 0\} = \partial U. \end{cases}$$

(Fully nonlinear) Solve example

$$\begin{cases} |Du| = 1 & \text{in } U = \{|x| > 1\} \subset \mathbb{R}^2 \\ u = x & \text{on } \partial U. \end{cases}$$

7.3. **Non-characteristic boundary data.** Let $x_0 \in \Gamma \subset \partial U$ and we are going to assume that

$$\Gamma \subset \{x_n = 0\}$$

in a neighborhood of x_0. This can be achieved by "flattening the boundary" in a neighborhood of x_0 which will transform the first order PDE into another first order PDE (with complicated x dependent coefficients but the same "linearity type").

We need to choose goo initial conditions for the characteristic ODEs

$$p(0) = p_0, \ z(0) = z_0 \ \text{ and } x(0) = x_0.$$

Clearly we should choose $z_0 = g(x_0)$. For p_0 the boundary conditions fix the tangential part, and the normal direction is fixed by the equation:

$$u(x', 0) = g(x') \ \text{ for } \ |x'| \ll 1$$

so

$$p_j^0 = \partial_j u(x_0) = \partial_j g(x_0) \ \text{ for } \ 1 \leq j \leq n-1.$$

The final condition is

$$F(p_0, z_0, x_0) = 0$$

provide n equations for the n unknowns in p^0, the only issue is the nonlinearity of the final equation. The choice of p^0 may (1) not exist, (2) exist but not be unique, (3) exists and is unique. Not only this but we need to solve for p^0 in a smooth way in a *neighborhood* of x_0 not just at x_0. This suggests we should be using the implicit function theorem.

Given (p_0, z_0, x_0) admissible at x_0 want to solve the following system for $|y - x_0| \ll 1$

$$\begin{cases} q^i(y) = \partial_i g(y) & (i = 1, \dots, n-1) \\ F(q(y), g(y), y) = 0 \end{cases}$$

with $q(y)$ smooth

Lemma 6. *There exists a unique solution q for $y \in \Gamma$ sufficiently close to x_0 if*

$$F_{p_n}(p^0, z^0, x^0) \neq 0.$$

Proof : Implicit function theorem.

General non-characteristic condition

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

Do some examples: $b(x) \cdot Du + c(x)u = 0$ transport equation give various scenarios.

Example: $\partial_{x_1} u = 0$ in $\mathbb{R}^2 \setminus B_1$ with boundary condition on ∂B_1.

7.4. **Local solvability.**

Lemma 7. *Assume that $F_{p_n}(p_0, z_0, x_0) \neq 0$. Then there is an open interval I containing 0 and a neighborhood W of x^0 in $\Gamma \subset \mathbb{R}^{n-1}$ and a neighborhood V of x^0 in $\mathbb{R}^n$ so that for each $x \in V$ there is a unique $s \in I$ and. a unique $y \in W$ so that*

$$x = \mathbf{x}(y, s).$$

The mapping $x \mapsto s, y$ are C^2.

Proof. We have

$$D\mathbf{x}(x^0, 0) = x^0$$

can get the result from inverse function theorem if

$$\det(D\mathbf{x})(x^0, 0) \neq 0.$$

Since $\mathbf{x}(y, 0) = y$ for $y \in \Gamma$

$$\mathbf{x}^j_{y_i}(x^0, 0) = \delta_{ij} \quad \text{for} \quad 1 \leq j \leq n-1$$

and

$$\mathbf{x}^n_{y_i}(x^0, 0) = 0.$$

The equation gives

$$\mathbf{x}^j_s(x^0, 0) = F_{p_j}(p^0, z^0, x^0) \quad \text{for} \quad 1 \leq j \leq n$$

so

$$D\mathbf{x}(x^0, 0) = \begin{bmatrix} \mathrm{id}_{n-1} & F_{p'}(p^0, z^0, x^0) \\ 0 & F_{p_n}(p^0, z^0, x^0) \end{bmatrix}$$

and the determinant is exactly $F_{p_n}(p^0, z^0, x^0) \neq 0$. ☐

Now we define

$$u(x) = z(\mathbf{y}(x), s(x))$$

Theorem 8. *The function u defined above is C^2 and solves*

$$F(Du(x), u(x), x) = 0 \quad \text{in} \quad V$$

with

$$u(x) = g(x) \quad \text{on} \quad \Gamma.$$

Proof. Somewhat involved computation, but this is exactly what we defined the characteristic equations to do. ☐

8. Conservation laws

We investigate nonlinear scalar conservation laws

$$\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = g(x). \end{cases}$$

Such equations arise as simplified models of gas dynamics. Method of characteristics applies, but generally speaking the solution defined by characteristics is only *local*, to extend to a global *discontinuous* solution we need to understand more about the nonlinear equation.

8.1. **Example of crossing characteristics.**

. Consider Burger's equation

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = \pm g(x) \end{cases}$$

with

$$g(x) = \begin{cases} 1 & x < -1 \\ -x & -1 < x < 0 \\ 0 & x > 0. \end{cases}$$

The characteristic equations identify the parameter s with t and

$$\dot{x} = z, \;\; \dot{z} = 0.$$

The equation is quasi-linear so no p ODE is needed.

Then

$$z(t) = z(0) = g(x_0) \;\; \text{and} \;\; x(t) = x_0 + g(x_0)t.$$

Draw picture and show characteristics etc.

8.2. **Weak solutions and Rankine-Hugoniot condition.**

We are stuck with discontinuous solutions so we need a way to "choose the right shock". The classical solution condition does not help us to specify because we are interested in discontinuous potential solutions. Need a notion of *weak solution.*

First multiply PDE by φ compactly supported in $\mathbb{R} \times [0, \infty)$ and integrate

$$0 = \int_{\mathbb{R}} \int_0^\infty (u_t + F(u)_x)\varphi(x,t)dtdx = -\int_{\mathbb{R}} u\varphi dx - \int_{\mathbb{R}\times[0,\infty)} u\varphi_t + F(u)\varphi_x \, dxdt.$$

So smooth solutions satisfy

$$0 = \int_{\mathbb{R}} u\varphi dx + \int_{\mathbb{R}\times[0,\infty)} u\varphi_t + F(u)\varphi_x \, dxdt$$

for *all* test functions $\varphi \in C_c^\infty(\mathbb{R} \times [0, \infty))$. We *define* a **weak solution** to be any bounded u satisfying this property.

Let's try to understand what the weak solution condition specifies via a simple scenario. Suppose that u is a weak solution on an open space-time region $V \subset \mathbb{R} \times (0, \infty)$ and u is smooth on either side a a smooth curve C, call V_ℓ the part of V left of C and V_r the part of V right of C. We are assuming that u is uniformly smooth up to the boundaries of V_ℓ and V_r, but the values and derivatives may jump along that curve.

First take a test function φ which has compact support in V_ℓ (or V_r) then integration by parts shows (using arbitrary φ in this class)

$$u_t + F(u)_x = 0 \ \text{ in the classical sense in } \ V_\ell \ \text{ and } \ V_r.$$

Now test with φ that does not necessarily vanish along C

$$\int_{\mathbb{R}\times[0,\infty)} u\varphi_t+F(u)\varphi_x \, dxdt = \int_{V_\ell} u\varphi_t+F(u)\varphi_x \, dxdt+\int_{V_r} u\varphi_t+F(u)\varphi_x \, dxdt$$

integrating by parts in each term on the right

$$\int_{V_\ell} u\varphi_t + F(u)\varphi_x \, dxdt = -\int_{V_\ell} (u_t + F(u)_x)\varphi \, dxdt + \int_C (u_\ell\nu^t + F(u_\ell)\nu^x)\varphi dS$$

where $\nu = (\nu^x, \nu^t)$ the the normal pointing outward from V_ℓ into V_r. Similarly from the right we get

$$\int_{V_\ell} u\varphi_t + F(u)\varphi_x \, dxdt = -\int_C (u_r\nu^t + F(u_r)\nu^x)\varphi dS$$

This leads to

$$0 = \int_C ((u_\ell - u_r)\nu^t + (F(u_\ell) - F(u_r))\nu^x)\varphi dS$$

for all test function φ so

$$(u_\ell - u_r)\nu^t + (F(u_\ell) - F(u_r))\nu^x = 0 \ \text{ on } \ C.$$

Now suppose we can parametrize C by $(\gamma(t), t)$ then $(\dot\gamma, 1)$ is a tangent vector pointing in the vertical direction so if we rotate 90 degrees clockwise we will get a normal vector pointing from left to right

$$(\nu^x(t), \nu^t(t)) = \frac{1}{\sqrt{1+|\dot\gamma|^2}}(1, -\dot\gamma)$$

which leads to

$$F(u_\ell) - F(u_r) = \dot\gamma(u_\ell - u_r).$$

We notate $[[F(u)]] = F(u_\ell) - F(u_r)$ the jump across the curve C and $\sigma = \dot\gamma$ the *speed* of the shock C then the **Rankine-Hugoniot condition** relates the shock speed to the jump values

$$\sigma = \frac{[[F(u)]]}{[[u]]}.$$

Examples: (1) Continue the previous example into a shock, (2) Riemann problem with upward jump (show region empty of characteristics, show shock and *rarefaction* possibilities, comment on the continuum of possible weak solutions).

8.3. **Entropy condition.** Recall that for a general scalar conservation law

$$u_t + F(u)_x = 0$$

the forward characteristics take the form

$$x(t) = x^0 + F'(g(x^0))t.$$

We make an assumption to rule out crossing of characteristics *backwards in time*. We call this an *entropy condition* in analogy with the thermodynamic idea that entropy increases, one could perhaps view this as a definition of the forward direction of time. There are more complicated formulations of the entropy condition which are more physically / mathematically principled but result in the same end result so for now we are just providing this simple end condition on the characteristics.

The entropy condition is only relevant on shocks where it requires

$$\sigma_\ell = F'(u_\ell) > \sigma > F'(u_r) = \sigma_r.$$

When F is uniformly convex the condition reduces simply to $u_\ell > u_r$ (draw picture with secant line and slopes).

Note: Weak solution plus entropy condition implies uniqueness, but there will always be *backwards non-uniqueness* (show example of Riemann data coming from moving shock or from condensing smooth data). Information is lost when a shock forms, smooth solutions do have backwards uniqueness.

Example: Indicator function of $[0, 1]$ as initial data.

8.4. **Riemann's problem.** We consider shock type initial data

$$g(x) = \begin{cases} u_\ell & x < 0 \\ u_r & x > 0 \end{cases}$$

for the conservation law

$$u_t + F(u)_x = 0.$$

Here F will be assumed to be uniformly convex and C^2.

Exploit the scaling invariance of the data/equation

$$\bar{x} = \lambda x \quad \text{and} \quad \bar{t} = \lambda t$$

if u solves RP then

$$u_\lambda(x, t) = u(\lambda x, \lambda t)$$

solves as well. We look for a *scaling invariant solution*

$$u_\lambda(x, t) = u_1(x, t) \quad \text{for all } \lambda > 0.$$

Taking $\lambda = t$ finds

$$u(x, t) = u(x/t, 1) = \bar{u}(\xi)$$

with $\xi = x/t$. Geometrically u is constant on space-time rays through the origin $x = \xi t$.

When $\bar{u}$ is continuous we call this a **centered rarefaction wave**.

Find the equation for $\bar{u}$

$$0 = u_t + F(u)_x = \bar{u}'\xi_t + F'(\bar{u})\bar{u}'\xi_x = -\frac{\xi}{t}\bar{u}' + F'(\bar{u})\bar{u}'\frac{1}{t}$$

so

$$(F'(\bar{u}) - \frac{\xi}{t})\bar{u}' = 0$$

that is *either*

$$\bar{u}'(\xi) = 0 \ \text{ or } \ F'(\bar{u}) = \xi$$

for each $\xi \in \mathbb{R}\backslash\{0\}$. If F' is invertible on an interval $[a, b]$ in $\mathbb{R}$, i.e. monotone, i.e. F convex, then this equation is invertible and generates a continuous rarefaction on that interval i.e. a solution

$$\bar{u}(\xi) = \begin{cases} (F')^{-1}(a) & \xi < a \\ (F')^{-1}(\xi) & a \leq \xi \leq b \\ (F')^{-1}(b) & x > b \end{cases}$$

Notice: when F' is monotone increasing this generates a continuous solution of a Riemann problem for $u_\ell = (F')^{-1}(a) < (F')^{-1}(b) = u_r$. When F' is monotone decreasing (i.e. F concave) this generates a continuous solution of a Riemann problem with $u_r > u_\ell$.

Example: Traffic flow model $0 \leq \rho \leq \rho_m$ is density, ρ_m is maximum density, traffic velocity is assumed to satisfy a law

$$v(\rho) = v_m(1 - \frac{\rho}{\rho_m}).$$

The *traffic flux* is

$$F(\rho) = \rho v(\rho) = v_m\rho(1 - \frac{\rho}{\rho_m})$$

and the corresponding conservation law for total mass is

$$\rho_t + F(\rho)_x = 0.$$

This flux is *concave* instead of convex. Shocks occur for upward jump initial data, rarefactions for downward jumps.

Examples: Red light turns green initial data ρ_m to the left, 0 to the right. Traffic jam ahead initial data, ρ_ℓ to the left ρ_m to the right.

9. Hamilton-Jacobi Equations

Hamilton-Jacobi equation IVP

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x). \end{cases}$$

This is a first order fully nonlinear PDE, the map $H(p) : \mathbb{R}^n \to \mathbb{R}$ is called the *Hamiltonian*, often we will assume that H is convex although initial discussion about characteristics does not require that.

Note that in $n = 1$ HJ is like an anti-derivative of a scalar conservation law. If $w = u_x$ then formally

$$w_t + H(w)_x = 0.$$

Hamilton-Jacobi equations generally will have shocks appear at the level of the derivative, we will still need a notion of weak solution to continue past these singularities. We will need again some "physical principle" to select the correct solution extended the one given by characteristics. This will come from a correspondence between the HJ equation and a problem of *optimal control*.

9.1. **Hamiltonian system of ODE.** Consider a Hamilton-Jacobi equation with $H(p, x)$ also x dependence in the Hamiltonian. A classic example from mechanics is $H(p, x) = \frac{1}{2}|p|^2 + V(x)$. Characteristic equations are

$$\begin{cases} X_t = D_p H(P, X) \\ P_t = -D_x H(P, X). \end{cases}$$

You may recognize this as Hamilton's formulation of classical mechanics.

Lagrangian formulation. Take $L(v, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ a given smooth function we call the *Lagrangian* or the *running cost*. We introduce the *action functional* or *cost functional* on paths $\gamma : [0, T] \to \mathbb{R}^n$

$$I[\gamma] = \int_0^T L(\dot{\gamma}, \gamma) dt.$$

Consider the problem of finding a path achieving the minimum

$$I[\mathbf{X}] = \min\{I[\gamma] : \gamma(0) = x_0, \ \gamma(T) = x_1, \ \gamma \in C^2([0, T])\}$$

with x_0 and x_1 some fixed points in $\mathbb{R}^n$. I.e. we are asking the question "what is the least cost path between x_0 and x_1".

Lemma 9 (Euler-Lagrange equations)**.** *If* $\mathbf{X}$ *minimizes above then* $\mathbf{X}$ *solves the ODE BVP*

$$-\frac{d}{dt}(D_v L(\dot{\mathbf{X}}, \mathbf{X})) + D_x L(\dot{\mathbf{X}}, \mathbf{X}) = 0 \quad on \quad 0 \le t \le T$$

with $\mathbf{X}(0) = x_0$ *and* $\mathbf{X}(T) = x_1$.

Proof. Standard method, $\frac{d}{\varepsilon}\Big|_{\varepsilon=0} I[\mathbf{X} + \varepsilon\varphi] = 0$ with $\varphi \in C_c^2([0, T])$. □

Example 1: $L(v, x) = \frac{1}{2}m|v|^2 - V(x)$

Example 2:

$$L(v, x) = \begin{cases} 0 & |v| \le 1 \\ +\infty & |v| > 1. \end{cases}$$

9.2. **Optimal control.** Now we consider a problem of *optimal control*, minimize the running cost with a payoff at the final time

$$u(x,t) = \inf\left\{\int_0^t L(\dot\gamma(s))ds + g(y) | \gamma(0) = x,\ \gamma(t) = y\right\}$$

The function u is called the *value function*, g is called the payoff or terminal cost. Optimizing trajectories still solve the same E-L equation but the boundary condition at the final time t is changed to:

$$D_v L(\dot{\mathbf{X}}(t)) + \nabla g(\mathbf{X}(t)) = 0.$$

Draw a space-time picture to conceptualize what is happening with the control problem, especially for T small.

Dynamic programming principle. The value function at time t_0 is the payoff for control from time $t > t_0$ to time t_0. (Again draw space-time picture).

Lemma 10 (DPP)**.**

$$u(x,t) = \inf\left\{\int_{t_0}^t L(\dot\gamma(s))ds + u(y,t_0) | \gamma(t_0) = y,\ \gamma(t) = x\right\}$$

Proof. Part 1. Fix y and let $\gamma_0 : [0,t_0] \to \mathbb{R}^n$ ε-optimal for $u(y,t_0)$. Then define concatenated path

$$\gamma = \gamma_0 + \gamma_1$$

for $\gamma_1 : [t_0, t] \to \mathbb{R}^n$ starting at y and ending at x. Then

$$u(x,t) \leq \int_0^t L(\dot\gamma(s))ds + g(\gamma(0)) \leq u(y,t_0) + \int_{t_0}^t L(\dot\gamma_1(s))ds + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, γ_1 was arbitrary ending at y, and then y was arbitrary we obtain one direction of the inequality.

Part 2. Let γ_* be ε-optimal for $u(x,t)$ then

$$\begin{aligned} u(x,t) &\geq \int_0^t L(\dot\gamma_*(s))ds + g(\gamma(0)) - \varepsilon \\ &= \int_0^{t_0} L(\dot\gamma_*(s))ds + g(\gamma_*(0)) + \int_{t_0}^t L(\dot\gamma_*(s))ds - \varepsilon \\ &\geq u(\gamma_*(t_0), t_0) + \int_{t_0}^t L(\dot\gamma_*(s))ds - \varepsilon \\ &\geq \inf\left\{\int_{t_0}^t L(\dot\gamma(s))ds + u(y,t_0) | \gamma(t_0) = y,\ \gamma(t) = x\right\} - \varepsilon \end{aligned}$$

□

Hamilton-Jacobi equation. If the value function u turns out to be C^1 then we can show that it solves the Hamilton-Jacobi equation. Note that u is globally defined irrelevant of characteristics and we will interpret it as a solution of the HJ equation even when it is not C^1.

Basically the HJ equation is the infinitesimal version of the dynamic programming principle. The following proof is formal for now

$$
\begin{aligned}
u(x,t+h) &= \inf\left\{\int_t^{t+h} L(\dot\gamma(s),\gamma(s))ds + u(y,t) | \gamma(t+h)=x,\ \gamma(t)=y\right\}\\
&\approx \inf\left\{\int_t^{t+h} L(v,x)ds + u(x-hv,t) | v\in\mathbb{R}^n\right\} + o(h)\\
&= u(x,t) + \inf_v\left[L(v,x)h + u(x-hv,t) - u(x,t)\right] + o(h)\\
&= u(x,t) + h\inf_v\left[L(v,x) - \nabla u(x,t)\cdot v\right] + o(h)\\
&= u(x,t) - h\sup_v\left[\nabla u(x,t)\cdot v - L(v,x)\right] + o(h)
\end{aligned}
$$

then we have showed formally sending $h\to 0$ that u satisfies a Hamilton-Jacobi equation

$$0 = u_t + \sup_v\left[\nabla u(x,t)\cdot v - L(v,x)\right] = u_t + L^*(\nabla u, x).$$

The formula appearing on the left is called the *Legendre-Fenchel transform.*

9.3. **Legendre-Fenchel Transform.** . Given $L:\mathbb{R}^n\to\mathbb{R}$ convex and superlinear

$$\lim_{|v|\to\infty}\frac{L(v)}{|v|} = +\infty$$

we define the *Legendre-Fenchel Transform*

$$L^*(p) = \sup_{p\in\mathbb{R}^n}\{p\cdot v - L(v)\}.$$

It will turn out that $(L^*)^* = L$ (when L is convex, otherwise it is the convex envelope). This gives a relation between a general Hamilton-Jacobi equation $u_t + H(Du,x) = 0$ and a control problem with running cost $L = H^*$. For this reason we will call $L^* = H$.

The pair L and $H = L^*$ are called *convex dual.*

Lemma 11. *The Hamiltonian H defined above is also convex and superlinear and $H^* = L$.*

Proof. Note that

$$\sup_{p\in\mathbb{R}^n}\{p\cdot v - L(v)\}$$

is a supremum of linear functions so it is convex.

Choose $v = \lambda p/|p|$ as a test minimizer to get

$$H(p)\geq \lambda|p| - \max_{B(0,\lambda)} L$$

Divide by $|p|$ and send $|p|\to\infty$ to get a lower bound on the growth.

Note

$$H(p) + L(v)\geq p\cdot v$$

for all $p, v \in \mathbb{R}^n$ and so

$$L(v) \geq \sup_p \{v \cdot p - H(p)\} = H^*(v).$$

On the other hand

$$H^*(v) = \sup_p \{v \cdot p - H(p)\} = \sup_p \{v \cdot p - \sup_q \{p \cdot q - L(q)\}\} = \sup_p \inf_q \{p \cdot (v-q) + L(q)\}$$

since L is convex it is above its tangent plane at v so

$$L(q) \geq L(v) + s \cdot (q-v)$$

for some s so

$$H^*(v) \geq \sup_p \inf_q \{p \cdot (v-q) + L(v) + s \cdot (q-v)\} \geq \inf_q \{s \cdot (v-q) + L(v) + s \cdot (q-v)\} = L(v).$$

□

Note that, assuming differentiability, the condition for $v_*(p)$ to optimize in

$$H(p) = \sup_{v \in \mathbb{R}^n} \{p \cdot v - L(v)\}$$

is

$$p = DL(v_*(p)).$$

That is $v_*(p)$ is the inverse of $DL(v)$.

In that case

$$H(p) = p \cdot v_*(p) - L(v_*(p))$$

and then

$$DH(p) = D_{v_*}(p \cdot v_*(p) - L(v_*(p)))D_p v_*(p) + v_*(p) = v_*(p).$$

So $DH(p)$ is the argument maximizer in the supremum, and DH and DL are inverses of each other.

Example: $L(v) = \frac{1}{2}|v|^2$. $H(p) = |p|$ (not superlinear).

9.4. **Hopf-Lax formula.**

Lemma 12. *Value function when $L(v, x) = L(v)$ and u_0 Lipschitz continuous satisfies the Hopf-Lax formula*

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ tL(\frac{x-y}{t}) + u_0(y) \right\}$$

Proof. Test with linear path for one direction, use Jensen's inequality for the other direction.

Proof that DPP implies solution of HJ equation is now rigorous at any differentiable point of u. We can show also that u is Lipschitz continuous and hence differentiable at almost every point. □

Combining with the Dynamic Programming Principle: for any $0 < s < t$

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ (t-s)L(\frac{x-y}{t-s}) + u(y,s) \right\}$$

Comparison principle for optimal control solution:

Lemma 13. *If $u_0 \geq v_0$ and u and v are corresponding value functions then $u(x,t) \geq v(x,t)$.*

Proof. Monotonicity of value function formula w.r.t. initial data. □

Time derivative bound:

Lemma 14. *For $h > 0$*

$$|u(x,t+h) - u(x,t)| \leq Ch$$

Proof. □

9.5. **Origins of Hamilton-Jacobi equations. Level set motion.** How to describe the motion of an interface with a PDE? For example let's consider the problem of an oriented moving interface Γ_t which moves by normal velocity

$$V_\nu(x) = c(x) \ \text{ for } \ x \in \Gamma_t$$

where $V_\nu(x)$ is the velocity of Γ_t in the outward normal direction. Let's suppose that we can represent $\Gamma_t = \partial\{u(x,t) > 0\}$ for some $u(x,t)$ called a *level-set function* or *level-set representation.* This is more flexible than assuming that Γ_t is a smooth graph, or a smooth embedding of an $n-1$-dimensional manifold because topological changes in Γ_t do not need to correspond to non-smoothness in u (show standard double well picture).

What does the equation for Γ_t say about u? If $X(t)$ is a path with $X(t) \in \Gamma_t$ for all $t > 0$ the equation says that

$$\dot{X}(t) \cdot \nu(X(t)) = c(X(t)).$$

Now consider, since $X(t) \in \Gamma_t$ for all t we have $u(X(t),t) \equiv 0$ so

$$0 = \frac{d}{dt} u(X(t),t) = \partial_t u + \dot{X}(t) \cdot Du(X(t),t).$$

Notice that since Γ_t is the zero level set of u we have $Du(x,t) = -\nu_{\Gamma_t}(x)|Du(x,t)|$. Thus

$$0 = \frac{d}{dt} u(X(t),t) = \partial_t u + \dot{X}(t) \cdot Du(X(t),t) = \partial_t u - \dot{X}(t) \cdot \nu_{\Gamma_t}(X(t))|Du| = u_t - c(X(t))|Du|.$$

Since $X(t)$ was arbitrary this leads to the equation

$$u_t = c(x)|Du| \ \text{ for } \ x \in \partial\{u(x,t) > 0\}.$$

There is no specification of u outside of its zero level set, but for simplicity we can impose that this Hamilton-Jacobi equation just holds everwhere and then it will certainly also hold on the zero level-set.

In general this leads to the formula for the *level-set velocity* of a function $u(x,t)$

$$V_\nu = \frac{u_t}{|Du|}.$$

Large deviations principles. Consider the heat equation

$$u_t = \frac{\varepsilon}{2}\Delta u \quad \text{with} \quad u(x,0) = \frac{1}{\alpha(n)}\mathbf{1}_{B(0,1)}(x).$$

Recall that we can interpret

$$u(x,t) = \mathbb{P}(x + W_{\varepsilon t} \in B(0,1))$$

where W_s is a standard Brownian motion started at 0.

When $x \notin B_1$ the event above is increasingly unlikely as $\varepsilon \to 0$. What we can compute is

$$v^\varepsilon(x,t) = \varepsilon \log u^\varepsilon(x,t).$$

This solves

$$v_t^\varepsilon = \varepsilon\frac{u_t^\varepsilon}{u^\varepsilon}, \ \nabla v^\varepsilon = \varepsilon\frac{\nabla u^\varepsilon}{u^\varepsilon}, \quad \text{and} \quad \Delta v^\varepsilon = \varepsilon\frac{\Delta u^\varepsilon}{u^\varepsilon} - \varepsilon\frac{|\nabla u^\varepsilon|^2}{(u^\varepsilon)^2} = \varepsilon\frac{\Delta u^\varepsilon}{u^\varepsilon} - \frac{1}{\varepsilon}|\nabla v^\varepsilon|^2$$

Thus

$$v_t^\varepsilon - \frac{\varepsilon}{2}\Delta v^\varepsilon = \varepsilon\left[\frac{u_t^\varepsilon}{u^\varepsilon} - \frac{\varepsilon}{2}\frac{\Delta u^\varepsilon}{u^\varepsilon}\right] + \frac{1}{2}|\nabla v^\varepsilon|^2 = \frac{1}{2}|\nabla v^\varepsilon|^2$$

resulting in the equation

$$v_t^\varepsilon = \frac{1}{2}|Dv^\varepsilon|^2 + \frac{\varepsilon}{2}\Delta v^\varepsilon \quad \text{with} \quad v^\varepsilon(x,0) = \begin{cases} 0 & \text{in } \ B(0,1) \\ -\infty & \text{in } \ \mathbb{R}^n \setminus B(0,1) \end{cases}$$

which is a *viscous Hamilton-Jacobi equation*. When $\varepsilon \to 0$ we expect to prove convergence to the Hopf-Lax solution of the HJ equation

$$v_t = \frac{1}{2}|Dv|^2 \quad \text{with} \quad v(x,0) = \begin{cases} 0 & \text{in } \ B(0,1) \\ -\infty & \text{in } \ \mathbb{R}^n \setminus B(0,1) \end{cases}$$

which, because the Hamiltonian is concave is a maximization problem

$$v(x,t) = \max_{y\in\mathbb{R}^n}\left\{v_0(y) - tL(\frac{x-y}{t})\right\} = \max_{y\in\mathbb{R}^n}\left\{v_0(y) - \frac{1}{2t}|x-y|^2\right\} = -\frac{1}{2t}d(x,B(0,1))^2$$

10. Similarity solutions

Just to give some idea on the interesting complications that can arise in nonlinear second order equations we will look briefly at the *porous medium equation*. In terms of techniques we will follow the idea of *similarity solutions*, i.e. identifying the scaling invariances of the equation and then looking for a scaling invariant solution. For linear equations this technique often led to identification of the fundamental solution. For nonlinear equations we cannot hope for quite that much, but scaling invariant solutions can still play a very important role in understanding solutions in general.

The porous medium equation arises in the modeling of fluid flow in porous media. In that setting $\rho : \mathbb{R}^n \to [0, \infty)$ is the mass density which satisfies the conservation law

$$\rho_t + \nabla \cdot (\rho u) = 0$$

where u is the fluid velocity. The *Darcy's Law* enforces a relationship between the fluid velocity and the pressure gradient

$$u = -\frac{1}{\nu} K \nabla p$$

where K is the permeability tensor, an $n \times n$ matrix, and ν is the fluid viscosity. We will just take $K = \mathrm{id}$ and $\nu = 1$ for simplicity. Then the pressure is determined by the *equation of state* (derived from some thermodynamic considerations)

$$p = \frac{1}{m-1} \rho^m \quad \text{with} \quad m > 1$$

(now I am just choosing non-dimensional mathematical constants to work out nicely). This leads to the *Porous Medium Equation*

$$\rho_t = \nabla \cdot (\rho \nabla p) = \nabla \cdot (\rho \rho^{m-1}) = \Delta(\rho^m).$$

First we look for a scaling invariance by considering

$$\rho_\lambda(x,t) = \lambda^\alpha \rho(\lambda^\beta x, \lambda t)$$

then

$$\partial_t \rho_\lambda = \lambda^{\alpha+1} \partial_t \rho, \ \nabla \rho_\lambda^m = \lambda^{m\alpha+\beta} \nabla \rho^m, \ \text{ and } \ \Delta(\rho_\lambda^m) = \lambda^{m\alpha+2\beta} \Delta(\rho^m)$$

so

$$\partial_t \rho_\lambda - \Delta \rho_\lambda^m = \lambda^{\alpha+1} \partial_t \rho - \lambda^{m\alpha+2\beta} \Delta \rho^m$$

which leads to the equation

$$\alpha + 1 = m\alpha + 2\beta.$$

Then we want to find a scaling invariant solution

$$\rho(x,t) = \lambda^\alpha \rho(\lambda^\beta x, \lambda t) \quad \text{for all} \quad \lambda > 0.$$

Taking $\lambda = 1/t$ we find

$$\rho(x,t) = t^{-\alpha} \rho(t^{-\beta} x, 1) = t^{-\alpha} v(t^{-\beta} x).$$

Then calling $y = t^{-\beta} x$ the PME becomes

$$\alpha v + \beta y \cdot Dv + \Delta v^m = 0$$

we then look for radial solutions $v(y) = w(|y|)$ and find

$$\alpha w + \beta r w' + (w^m)'' + \frac{n-1}{r} (w^m)' = 0.$$

Now if we set $\alpha = n\beta$, which together with the previous constraint implies

$$\alpha = \frac{n}{(m-1)n+2} \quad \text{and} \quad \beta = \frac{1}{(m-1)n+2}$$

then this simplifies to

$$(r^{n-1}(w^m)')' + \beta(r^n w)' = 0$$

integrating this and using zero boundary conditions at infinity

$$r^{n-1}(w^m)' + \beta r^n w = 0$$

or

$$(w^m)' = -\beta r w$$

which, if $w > 0$, we can divide by w to get,

$$(w^{m-1})' = -\frac{m-1}{m}\beta r$$

so

$$w^{m-1} = b - \frac{m-1}{2m}\beta r^2 \ \text{ if } \ w > 0.$$

Thus we find the formula

$$w(r) = (b - \frac{m-1}{2m}\beta r^2)_+^{\frac{1}{m-1}}$$

plugging back in to the original formulation we find

$$\rho(x,t) = \frac{1}{t^\alpha}(b - \frac{m-1}{2m}\beta\frac{|x|^2}{t^{2\beta}})_+^{\frac{1}{m-1}}$$

with

$$\alpha = \frac{n}{(m-1)n+2} \ \text{ and } \ \beta = \frac{1}{(m-1)n+2}.$$

This special solution is called the Barenblatt solution. (Draw picture). Note that the positivity set has a finite speed of propagation, which is totally unlike the linear heat equation. It plays a somewhat analogous role to the fundamental solution, except that the equation is nonlinear so there is no convolution formula which relates general solutions linear combinations of Barenblatt profiles. Nonetheless the Barenblatt profile still plays a universal role, in an appropriate limit it describes the long time asymptotic behavior of arbitrary compactly supported non-negative initial profiles. Also the solution is obviously not a classical solution, nonetheless we we want to regard it as the correct solution of the nonlinear equation.

11. Weak derivatives and Sobolev spaces

To proceed further with the existence theory of general second order equations and boundary value problems it becomes necessary to get the functional analytic framework right.

11.1. **Weak derivatives.** We discussed the notion of distributional derivative already in the previous semester. The idea is to define derivative by duality/integration by parts against smooth test functions. Given a locally integrable function u we say that a distribution f is $f = D^\alpha u$ if

$$\langle f, \varphi \rangle = (-1)^{|\alpha|} \int_{\mathbb{R}^n} u(x) D^\alpha \varphi(x)\ dx$$

for all test functions $\varphi \in C_c^\infty(U)$.

This formula always defines s distribution, we additionally say that f is the α-th *weak derivative* of u if f is represented by a locally integrable function i.e.

$$\int_{\mathbb{R}^n} f(x)\varphi(x)\ dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} u(x) D^\alpha \varphi(x)\ dx$$

for all test functions $\varphi \in C_c^\infty(U)$. Actually, by a density argument, this also will hold for just $C_c^{|\alpha|}(U)$ functions.

The uniqueness of the weak derivative is straightforward, use the formula to show

$$\int_U (f(x) - \tilde{f}(x))\varphi(x)\ dx = 0 \ \text{ for all } \ \varphi \in C_c^\infty(U).$$

Examples: $|x|^r$ has weak derivatives of all orders $m > 0$ so that $r - m > -n$ (required for local integrability). For example the Laplace fundamental solution $|x|^{2-n}$ has weak derivatives of order 1 but not 2 (as we know the Laplacian is a δ-distribution which is not represented by a locally integrable function).

The *Sobolev space* $W^{k,p}$ is defined for $1 \le p \le \infty$ and $k \in \mathbb{N}$

$$W^{k,p}(U) = \{u \in L^1_{loc}(U) : \ D^\alpha u \in L^p(U) \ \text{ for all } \ |\alpha| \le k\}.$$

The Sobolev norm is

$$\|u\|_{W^{k,p}(U)} = \sum_{|\alpha| \le k} \left(\int_U |D^\alpha u|^p\ dx \right)^{1/p}$$

and

$$\|u\|_{W^{k,p}(U)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^\infty(U)}.$$

The L^2-based Sobolev spaces have an inner product structure and are often call $H^k = W^{k,2}$.

Example: Let x_k be a countable dense subset of U and define

$$\sum_k 2^{-k} |x - x_k|^{-r}$$

this is in $W^{1,p}$ for $r < \frac{n-p}{p}$.

We call $W_0^{k,p}(U)$ to be the closure of $C_c^\infty(U)$ in the $W^{k,p}(U)$ norm. Unlike for the $k = 0$ case (i.e. L^p spaces) this will actually be a nontrivially distinct subspace of $W^{k,p}$, this is good news because we can meaningfully encode boundary conditions within the Sobolev context. To be precise, one can show, the *boundary trace map* map $T_{\partial U}\phi = \phi|_{\partial U}$ extends from the smooth functions to a continuous linear mapping (with respect to the $W^{k,p}$ norm) into $W^{k-1,p}(\partial U)$.

Nontrivial results: $C_c^\infty(U)$ is dense in $W^{k,p}(V)$ for an V compactly contained in U, If ∂U is smooth then $C^\infty(\overline{U})$ is dense in $W^{k,p}(U)$.

Theorem 15. *Assume U is bounded and ∂U is C^1. Then there is a bounded linear operator $T : W^{1,p}(U) \to L^p(\partial U)$*

$$\|Tu\|_{L^p(\partial U)} \leq C(p, U)\|u\|_{W^{k,p}(U)}$$

such that

$$Tu = u|_{\partial U} \quad \textit{for} \quad u \in W^{1,p}(U) \cap C(\overline{U})$$

Proof. Using a smooth partition of unity can decompose $u = \sum \zeta_j u$ in components which are supported in local neighborhood where ∂U is a local graph. We just do the case when $\Gamma = \partial U \cap B = \{x_n = 0\} \cap B$ for some ball B and u is zero on ∂B:

$$\int_\Gamma |u|^p dx' \leq \int_{x_n=0} \zeta |u|^p dx' = -\int_{B_+} (\zeta|u|^p)_{x_n}\, dx = -\int_{B_+} |u|^p \zeta_{x_n} + p|u|^{p-1}(\operatorname{sgn}(u)) u_{x_n} \zeta\, dx$$

If ∂U is not flat do a standard change of variables to flatten the boundary. □

Theorem 16. *If U has C^1 boundary then $u \in W_0^{1,p}(U)$ if and only if $Tu = 0$.*

11.2. **Weak formulation of second order elliptic equations.** Consider the PDE operators

$$Lu = -\nabla \cdot (A(x)\nabla u) + b(x) \cdot \nabla u + c(x)u$$

and

$$L'u = -\mathrm{Tr}(A(x)D^2 u) + b(x) \cdot \nabla u + c(x)u.$$

We will assume $A, b, c \in L^\infty(U)$. These operators are called elliptic if

$$\xi \cdot A(x)\xi \geq \theta|\xi|^2.$$

The first operator is said to be in divergence form, the second is said to be in non-divergence form. Although there are possible transformations between these forms it is often useful to consider them separately, divergence form equations are well suited to integration by parts / variational techniques, and non-divergence form operators are well suited to maximum principle based techniques.

We will consider the Dirichlet problem

$$Lu = f + \nabla \cdot h \;\text{ in }\; U \;\text{ and }\; u = 0 \;\text{ on }\; \partial U$$

with $f, h \in L^2$. It turns out that this problem is very naturally posed in the space $H_0^1(U)$. In fact the trace theorem means that the boundary data is naturally included already in the space. However the operator Lu cannot obviously be evaluated on H^1_0 functions since it involves a second derivative.

Here is where the weak formulation comes in, if u were a smooth solution then multiplying by a test function $\varphi Lu = 0$ pointwise in U and integrating on U we would find

$$B[u,\varphi] = \int_U A(x)\nabla u \cdot \nabla \varphi + b(x)\cdot \nabla u \varphi + c(x) u \varphi \; dx = \int_U f\varphi + h \cdot \nabla \varphi \; dx.$$

This bilinear form is well defined on $H^1_0(U)$. So we say u is a variational solution of the PDE BVP if

$$B[u,\varphi] = \int_U f\varphi + h\cdot \nabla\varphi \; dx \;\text{ for all }\; \varphi \in H^1_0(U).$$

Other boundary conditions can be studied as well, if g is the trace of $G \in H^1(U)$ then $w = u - G \in H^1_0(U)$ and solves

$$Lw = f - LG$$

note that the right hand side fits into the form established previously.

The existence of a solution to this problem would follow from the Riesz Representation theorem if B were an alternative inner product on H^1_0. This would require symmetry of B which we do not have, nonetheless there is an alternative theorem which applies.

Theorem 17 (Lax-Milgram)**.** *Assume $B : H \times H \to \mathbb{R}$ where H is a real Hilbert space and B is bilinear satisfying (1) boundedness*

$$|B[u,v]| \leq C\|u\|\|v\|$$

and coercivity

$$c\|u\|^2 \leq B[u,u]$$

then for f any bounded linear functional on H there is a unique $u \in H$ so that

$$B[u,v] = \langle f, v\rangle \;\text{ for all }\; v \in H.$$

11.3. **Variational problem.** Remark that divergence form equations arise in minimization of energy functionals like

$$I[u] = \int_U \frac{1}{2} A(x)\nabla u \cdot \nabla u - f(x)u - h(x)\cdot \nabla u \; dx$$

for example over the admissible class

$$\mathcal{A} = H^1_0(U).$$

The space $H^1(U)$ is very natural for the energy functional I.

Recall the Rayleigh quotient idea for the ground state eigenvalue of a linear differential operator

$$\lambda_0(U;A) = \inf_{u \in H^1_0(U)} \frac{\int_U \frac{1}{2}A(x)\nabla u \cdot \nabla u \; dx}{\|u\|^2_{L^2}}.$$

With $A(x) = id$ the eigenvalues $\lambda_0(U)$ is the inverse of the Poincaré constant

$$\|u\|_{L^2(U)} \leq \frac{1}{2}\lambda_0(U)^{-1/2}\|\nabla u\|_{L^2(U)} \quad \text{for all} \quad u \in H^1_0(U).$$

Note that $\lambda_0(\cdot)$ is monotone decreasing with respect to domain containment because $u \in H^1_0(U)$ is also in $H^1_0(V)$ for any $V \supset U$. This means that

$$\lambda_0(U)^{-1/2} \leq \lambda_0(B_R)^{-1/2}$$

where R is the infimal radius so that $U \subset B_R(y)$ for some y. The Dirichlet eigenvalue of a ball can be computed explicitly by separation of variables in polar coordinates and scales like

$$\lambda_0(B_R) = R^{-2}\lambda_0(B_1).$$

Another quick application of the Poincaré inequality

$$u_t - \Delta u = 0 \ \text{ in } \ U \ \text{ with } \ u = 0 \ \text{ on } \ \partial U$$

then

$$\frac{d}{dt}\int u^2\, dx = -2\int_U |Du|^2\, dx \leq -\lambda_0(U)\int_U u^2\, dx$$

which, by Gronwall, gives

$$\int_U u^2\, dx \leq \int_U u_0^2\, dx e^{-\lambda_0(U)t}.$$

Note that this is sharp because the ground state eigenvector $-\Delta\varphi_0 = \lambda_0\varphi_0$ gives a solution

$$v(x,t) = e^{-\lambda_0 t}\varphi_0.$$

11.4. **Method of eigenfunctions.**

Theorem 18. *Smooth domain U in $\mathbb{R}^n$, $-\Delta$ has a sequence of Dirichlet eigenvalues (w/ multiplicity)*

$$0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

which accumulate only at $+\infty$ and there is a corresponding sequence of eigenfunctions $\varphi_k \in H^1_0(U)$ which solve (in the weak sense)

$$-\Delta\varphi_k = \lambda_k\varphi_k$$

and $(\varphi_k)_{k=1}^\infty$ form a basis for $L^2(U)$.

See chapter 6 of Evans for proofs.

Eigenfunction expansion is the generalization of Fourier methods to boundary value problems. For example we can represent a solution of the heat equation $u_t - \Delta u = 0$

$$u(x,t) = \sum_k a_k e^{-\lambda_k t}\varphi_k(x)$$

or the wave equation $u_{tt} - \Delta u = 0$

$$u(x,t) = \sum_k a_k e^{i\lambda_k^{1/2}t}\varphi_k(x).$$

Can think term by term to gain some intuition about what these equations are doing in the eigenfunction/Fourier basis.

11.5. **Galerkin methods.** Project the variational problem onto a finite dimensional subspace of H^1. For example if we are trying to solve for $u \in H^1_0(U)$ with

$$\int_U \nabla u \cdot \nabla \phi dx = \int_U f\phi \ dx \ \text{ for all } \ \phi \in H^1_0(U)$$

we can choose some finite linearly independent collection w_j for $1 \le j \le N$ and then look for $v \in \mathrm{span}(w_1, \ldots, w_j)$ so that

$$\int_U \nabla v \cdot \nabla w_i dx = \int_U f w_i \ dx \ \text{ for all } \ 1 \le i \le N$$

or writing $v = \sum v_i w_i(x)$

$$\sum_j \int_U \nabla w_i \cdot \nabla w_j dx v_j = \int_U f w_j \ dx \ \text{ for all } \ 1 \le j \le N$$

which is an $N \times N$ linear system. If w_j were an orthonormal basis of eigenfunctions the matrix would be identity. For practical purposes, if one did not know the eigenfunctions, there are many other choices of bases.

12. Review

- **The Laplace equation:** Harmonic functions, mean value theorems, maximum principles, energy minimization; Fundamental solution; Boundary value problems; Green's functions.
- **Diffusion:** The one-dimensional diffusion equation; Uniqueness: integral methods and maximum principles; Fundamental solution and the global Cauchy problem; Random walks; Global Cauchy problem, maximum principles; Energy methods; Some nonlinear problems: traveling waves.
- **Waves and vibrations:** General concepts, e.g., types of waves, group velocity, dispersion relations; One-dimensional wave equation, waves on a string; The D'Alembert formula and characteristics; Classification of second-order linear equations; Multi-dimensional wave equation, the Cauchy problem; Energy methods / uniqueness.
- **First order equations: Scalar conservation laws and Hamilton-Jacobi Equations:** Linear transport equation and conservation laws; Method of Characteristics; Weak solutions and shock waves; Entropy solutions; Hamilton-Jacobi Equations; Lagrangian-Hamiltonian duality; Control formulation and dynamic programming principle; viscosity solutions.
- **Variational formulation of elliptic problems:** Sobolev spaces, trace theorem, Linear operators and duality; Lax–Milgram theorem and minimization of bilinear forms; Variational formulation of Poisson's equation in higher dimensions.