1. Basics

Stein and Shakarchi Ch 1

1.1. **Complex plane.** Real and Imaginary parts, complex addition, multiplication, geometry, triangle inequality, complex conjugate, \( \text{Re}(z) = \frac{z + \bar{z}}{2} \)
etc., complex magnitude, \( 1/z = \bar{z}/|z|^2 \), polar form \( z = re^{i\theta} \) where \( r = |z| \), \( \theta = \arg(z) \), and Euler’s formula \( e^{i\theta} = \cos \theta + i \sin \theta \), complex multiplication as homothety.

1.2. **Topology of \( \mathbb{C} \).** Notion of convergence, Cauchy sequences, \( \mathbb{C} \) is complete, notation for discs \( D_r(z_0) \), circles \( C_r(z_0) \), unit disk \( D_1 \), unit circle \( \partial D_1 \), open, closed, bounded, compact (closed and bounded iff sequentially compact iff open cover compact ... nested intersection property may be useful later), connected set, equivalence of connected and path connected for open sets in \( \mathbb{C} \).

1.3. **Holomorphic functions.** Continuity (same as for functions on \( \mathbb{R}^2 \)), Holomorphic at \( z_0 \) if the difference quotients converge, limit is called the derivative, function is holomorphic on \( \Omega \) open if it is complex differentiable at every point, on \( \mathbb{C} \) closed if there is \( \Omega \) open containing \( C \) on which it is holomorphic, on all of \( \mathbb{C} \) entire. \( f(z) = z \) and any polynomial \( p(z) = a_0 + a_1z + \cdots + a_nz^n \). \( f(z) \) is holomorphic on \( \mathbb{C} \setminus \{0\} \) and \( f'(z) = -1/z^2 \). \( f(z) = \bar{z} \) is NOT holomorphic. Useful also to write differentiability as

\[
\psi(h) = \begin{cases} 
\frac{f(z_0 + h) - f(z_0) - ah}{h} & \text{if } h \neq 0 \\
0 & \text{if } h = 0
\end{cases}
\]

where \( \psi \) is defined for small enough \( |h| \) and \( \psi(h) \to 0 \) as \( h \to 0 \). Also can write this as \( o(h) \) (with this as the precise defn). Proposition: \((f + g)' = f' + g', (fg)' = f'g + g'f\), if \( g(z_0) \neq 0 \) then \((f/g)' = (f'g - g'f)/g^2\), if \( f : \Omega \to U \) and \( g : U \to \mathbb{C} \) are holomorphic then \((g \circ f)'(z) = g'(f(z))f'(z)\) for \( z \in \Omega \).

In particular polynomials are holomorphic and complex derivatives have the known formulae.

1.4. **Cauchy-Riemann equations.** Compute \( f'(z) \) two ways with \( h = h_1 + ih_2 \) with \( h_2 \in \mathbb{R} \), and \( h = ih_2 \) with \( h_2 \in \mathbb{R} \), find \( \partial_x f = \frac{1}{i} \partial_y f \) and

\[
\partial_x u = \partial_y v \quad \text{and} \quad \partial_y u = -\partial_x v.
\]
Also note that $\partial^2_x u = \partial_y(-\partial_y u) = -\partial^2_y u$ so $u$ is harmonic (assuming sufficient regularity for now).

Define the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$
and

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$$

Then if $f$ is holomorphic then

$$\frac{\partial f}{\partial \bar{z}} = 0$$
and

$$f'(z) = \frac{\partial f}{\partial z} = 2\frac{\partial u}{\partial z}$$

Also jacobian formula for $F(x,y) = (u(z),v(z))$

$$\det(DF(x,y)) = |f'(z)|^2$$

Note that sign is always positive so $Df$ is orientation preserving, anti-holomorphic functions reverse orientation.

CONVERSE: If $u$ and $v$ are $C^1$ and satisfy CR equations then $f(z) = u + iv$ is holomorphic.

1.5. **Power series.** $e^z = \sum z^n/n!$, note by triangle ineq $|e^z| \leq e^{|z|}$ which implies absolute summability for all $z \in \mathbb{C}$. Geometric series $\sum^{N} z^n = \frac{1-2^{N+1}}{1-z}$ converges in $|z| < 1$ to the function $1/(1- z)$ holomorphic in $\mathbb{C} \setminus \{1\}$.

Note that if power series converges absolutely at $z_0$ then it also converges absolutely for all $z$ with $|z| \leq |z_0|$.

Theorem: Given series $\sum a_n z^n$ there is $0 \leq R \leq \infty$ called radius of convergence so that for $|z| < R$ power series converges absolutely, for $|z| > R$ diverges, and formula

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

Proof: Call $L = 1/R$ and let $\varepsilon > 0$ so that $(L + \varepsilon)|z| = r < 1$

$$|a_n||z|^n = (|a_n|^{1/n}|z|)^n \leq ((L + \varepsilon)|z|)^n = r^n$$

for $n$ sufficiently large. Since this geometric series is absolutely summable we get convergence.

If $|z| > R$ similar argument shows that there is a subsequence of terms so that $|a_n||z|^n \to \infty$ which means that the partial sums cannot be Cauchy and hence cannot converge.

Trigonometric functions

$$\cos z = \sum (-1)^n z^{2n}/(2n)! \quad \text{and} \quad \sin z = \sum (-1)^n z^{2n+1}/(2n+1)!$$

and

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \quad \text{and} \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

also reverse formulae $e^{iz} = \cos z + i\sin z$ and $e^{-iz} = \cos z - i\sin z$.

Note for a function given by a power series $f(z) = f(\bar{z})$. 

Differentiating term by term: Note if \( f_n(z) \to f(z) \) uniformly this does not generally imply that the derivatives \( f'_n(z) \to f(z) \) (at least in real variable problems!). For power series it is true inside the radius of convergence. If we establish that \( \frac{f_n(z) - f_n(z_0)}{z - z_0} \) converges to \( \frac{f(z) - f(z_0)}{z - z_0} \) uniformly then the result is true.

Apply to power series using \( a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}) \)

\[
|\sum_{N} a_n \frac{(z_0 + h)^n - z_0^n}{h}| \leq \sum_{N} n|a_n||z_0|^{n-1}
\]

this is \( \frac{n}{|z_0|} |a_n||z_0|^n \) but the coefficient \( b_n = \frac{n}{|z_0|} |a_n| \) has the same radius of convergence since

\[
\limsup b_n^{1/n} = \limsup n^{1/n}|z_0|^{-1/n}|a_n|^{1/n} = 1/R.
\]

Corollary: Power series are infinitely differentiable, derivatives are the termwise derivative power series, and the radii of convergence of all derivatives are the same.

Can also center power series at \( z_0 \). A function is called analytic at \( z_0 \) if it has a power series representation at \( z_0 \) in a positive radius neighborhood. It turns out that holomorphic and analytic on an open domain \( \Omega \) are equivalent!

1.6. Integration on curves. It is useful to distinguish between the one-dimensional objects with orientation in \( \mathbb{C} \) and their parametrizations.

Parametrized curve is a map \( z(t) : [a,b] \to \mathbb{C} \), called smooth if \( z \) is \( C^1 \) on \( [a,b] \) and \( z'(t) \neq 0 \) (one-sided difference quotients at \( a,b \)). Recall issue with \( z' = 0 \). Called piecewise smooth if \( z \) is continuous and \( [a,b] \) can be partitioned into (almost) disjoint closed subintervals on which \( z \) is smooth.

Two parametrized curves \( z \) on \( [a,b] \) and \( w \) on \( [c,d] \) are equivalent if there is a strictly monotone increasing and smooth change of variables \( t : [c,d] \to [a,b] \) and

\[
w(s) = z(t(s)).
\]

The condition \( t'(s) > 0 \) guarantees that \( z \) and \( w \) have the same orientation.

An equivalent family of smooth parametrized curves is an (oriented) curve \( \gamma \) in \( \mathbb{C} \). Piece-wise smooth curves are defined similarly. \( \gamma^- \) is the same curve with reverse orientation, can also write \( -\gamma \).

Endpoints of curve are \( z(a) \) and \( z(b) \) (independent of parametrization). Curve is closed if endpoints are the same. Curve is simple if no self-intersections.

For example \( C_r(z_0) \) is an un-oriented curve a positive (counter-clockwise) parametrization is

\[
z(t) = z_0 + re^{it} \quad t \in [0,2\pi]
\]

negative (clockwise) oriented parametrization

\[
z(t) = z_0 + re^{-it} \quad t \in [0,2\pi].
\]
The integral on a piece-wise smooth curve is
\[ \int_{\gamma} f(z) \, dz = \int_{a}^{b} f(z(t))z'(t) \, dt \]

where we need to justify that the definition does not depend on choice of parametrization: with \( w(s) = z(t(s)) \) as before \( dt = t'(s) \, ds \) so change of variables formula gives
\[ \int_{a}^{b} f(z(t))z'(t) \, dt = \int_{c}^{d} f(w(s))z'(t(s))t'(s) \, ds = \int_{c}^{d} f(w(s))w'(s) \, ds. \]

The length of a piecewise smooth curve is defined
\[ \text{length}(\gamma) = \int_{a}^{b} |z'(t)| \, dt = \sup_{P} \sum_{i=0}^{N} |z(t_{i+1}) - z(t_{i})|. \]

Path length integral
\[ \int_{\gamma} f(z) |z| \, dz = \int_{a}^{b} f(z(t)) |z'(t)| \, dt \]

Path integration is linear, changes sign under orientation reversal, natural bound
\[ |\int_{\gamma} f(z) \, dz| \leq \sup_{\gamma} |f| \, \text{length}(\gamma). \]

More generally
\[ |\int_{\gamma} f(z) \, dz| \leq \int_{\gamma} |f(z)| \, |dz| \]

1.7. Primitives. A function \( f \) on a domain \( U \) has a primitive \( F \) if \( F \) is holomorphic on \( U \) and \( F'(z) = f(z) \).

Lemma: If \( F \) is a primitive for \( f \) continuous then
\[ \int_{\gamma} f(z) \, dz = F(w_{2}) - F(w_{1}) \]
for any curve \( \gamma \) beginning at \( w_{2} \) and ending at \( w_{1} \).

In particular if \( f \) has a primitive on \( U \) then
\[ \int_{\gamma} f \, dz = 0 \]
for any closed curve \( \gamma \) in \( U \).

Note: NOT every holomorphic function on a domain \( U \) has a primitive on \( U \), e.g. \( 1/z \) on \( \mathbb{D} \setminus \{0\} \).
1.8. **Goursat’s theorem.** If $\Omega$ is an open set in $\mathbb{C}$, $f$ is holomorphic in $\Omega$ and $T$ is a triangle whose interior is also contained in $\Omega$ then

$$\int_T f(z) \, dz = 0$$

We use the following property of compact sets: if $K_1 \supset K_2 \supset \cdots$ is a nested collection of compact sets then $\bigcap K_j$ is nonempty. If the diameter $\text{diam}(K_j) \to 0$ then the intersection is a singleton.

Remark: the proof technique also works for rectangles but it is easier to derive the rectangle case from the triangle case than vice versa.

Derive the same result for rectangles by splitting into two triangles.

Idea: reduce to zooming in, holomorphic functions look locally linear for zoomed in case, linear functions have a primitive and satisfy the theorem.

Why is it not true for $C^1$ functions.

1.9. **Local existence of primitives.** A holomorphic function on an open disc has a primitive on that disc.

Proof: Define primitive by integrating from 0 to $z$ along right angle path $\gamma_z$. Use Goursat to write $F(z + h) - F(z) = \int_\eta f(z) \, dz$ where $\eta$ is the line segment from $z$ to $z + h$. Use $f$ continuous write $f(w) = f(z) + \psi(w)$ and compute.

Corollary (Cauchy’s theorem) The integral of a holomorphic function in a disc on any closed curve in that disc is zero.

1.10. **Homotopy.** Two parametrized curves in a set $U$ are homotopic if there is a map $\psi(t, s) : [a, b] \times [0, 1] \to U$ continuous in $s$ and piecewise smooth in $t$ for each $s$ so that

$$\psi(t, 0) = \gamma \quad \text{and} \quad \psi(t, 1) = \eta.$$ 

Describe as continuous deformation of curves, draw pictures (1) two paths, (2) circle around missing point.

For non-closed paths we will include in the definition that homotopy fixes end points. For closed paths we will include in the definition that homotopy preserves the property of being closed.

Definition generalizes to oriented curves, creates an equivalence relation on curves in $U$.

Homotopy form of Cauchy’s theorem: if $f$ holomorphic in $U$ and $\gamma$ and $\eta$ are homotopic in $U$ then

$$\int_\gamma f(z) \, dz = \int_\eta f(z) \, dz$$

I.e. contour integral is homotopy invariant, constant on homotopy equivalence classes. Proof: 1. do case of nearby curves, 2. reduce to case of nearby curves by the homotopy.

Note: If $\gamma$ is a closed curve homotopic to a point then $\int_\gamma f(z) \, dz = 0$.

Define $U$ is simply connected if it is connected and every closed curve is homotopic to a point. Non-examples: annulus, set minus several points.
If $U$ is simply connected then $\int_{\gamma} f(z)dz = 0$ for every closed curve $\gamma$ contained in $U$.

Examples: Convex implies simply connected.

Holomorphic functions on simply connected domains have a primitive: define primitive by integrating on arbitrary path from fixed base point, use Cauchy’s theorem to show the definition is good.

1.11. **Cauchy's integral formula.** Statement $f$ holomorphic in $D(z_0, r)$ then

$$f(z) = \frac{1}{2\pi i} \int_{C} \frac{f(\zeta)}{\zeta - z} d\zeta.$$  

- Note $f(\zeta)/(\zeta - z)$ not holomorphic on the inside of disc so no contradiction.
- Keyhole contour with parameters $\varepsilon > 0$ and $\delta > 0$. Fix $\varepsilon > 0$ and send $\delta \to 0$ to show equality of outer and inner integrals.
- Write
  $$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta) - f(z)}{\zeta - z} + \frac{f(z)}{\zeta - z} $$

and use holomorphicity so first term converges to $f'(z)$ as $\zeta \to z$.

Holomorphic implies infinitely many complex derivatives and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C} \frac{f(z)}{(\zeta - z)^{n+1}} d\zeta.$$  

**Proof:** If $f$ holomorphic in a neighborhood of $z_0$ can choose a circle $C_r(z_0)$ so that $f$ is holomorphic in neighborhood of $C$. For $z$ in that disc take Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_{C} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) = \frac{1}{2\pi i} \int_{C} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)} d\zeta = \frac{1}{2\pi i} \int_{C} \frac{f(\zeta) - f(z)}{\zeta - z_0} d\zeta = \frac{1}{2\pi i} \int_{C} \frac{f(\zeta) - f(z)}{\zeta - z_0} \left(1 - \frac{z - z_0}{\zeta - z_0}\right) d\zeta$$

and expand the geometric series.

**COROLLARY OF PROOF:** If $f$ is holomorphic in $D_r(z_0)$ then the power series expansion for $f$ centered at $z_0$ has radius of convergence at least $r$.

In particular holomorphic in an open set $U$ is equivalent to analytic in $U$.

**COROLLARY:** If $f$ has a primitive in a disc $D_r(z_0)$ then $f$ is holomorphic in $D_r(z_0)$

**COROLLARY:** (Morera’s theorem) If $\int_{T} f(z) dz = 0$ for all triangles $T$ in a disc $D_r(z_0)$ then $f$ has a primitive and is holomorphic in $D_r(z_0)$.

Cauchy inequalities

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi R^n} \sup_{D_r(z)} |f|$$

Liouville Theorem- bounded entire implies constant by Cauchy inequality for $n = 1$.

Every non-constant polynomial has a root in $\mathbb{C}$: if not $1/P$ would be a bounded holomorphic function (since polynomials go to $\infty$ at $\infty$).
Every degree $n \geq 1$ polynomial has exactly $n$ roots in $\mathbb{C}$ and $P(z) = a_n(z - z_1) \cdots (z - z_n)$.

proof: $P$ has a root $z_1$ then consider expanding each term by binomial formula

$$P(z) = P(z - z_1) = b_n(z - z_1)^n + b_{n-1}(z - z_1)^n + \cdots + b_1(z - z_1) + b_0$$

with $b_n = a_n$, must be $b_0 = 0$ since $P(z_1) = b_0 = 0$. Thus $P(z) = (z - z_1)Q(z)$ where $Q$ has degree $n - 1$. Induction.

If $f$ holomorphic on connected $\Omega$ vanishes on a set $E$ with an accumulation point then $f$ is zero. (More generally, unique continuation, $f$ is determined by its values on any set with an accumulation point, just think about $f-g$ for any two functions with the same values on $E$).

proof: 1. Show that if $z_0 = 0$ is an accumulation point of $\{f = 0\}$ then $f = 0$ in a neighborhood of $z_0$. 2. Consider $U = \{f = 0\}^c$ this is open by definition, but also closed by the part 1, thus $U$ is open and closed in $\Omega$ and $f$ is constant in $\Omega$ (connected).

Proof of 1: Power series expansion in $D_r(z_0)$, if all coefficients are zero done. Otherwise

$$f(z) = a_m(z - z_0)^m(1 + g(z)) \quad \text{with} \quad a_m \neq 0$$

where $g(z) \to 0$ as $z \to 0$. But then $|f(z)| \geq |a_m|/2|z - z_0|^m$ on sufficiently small disc around $z_0$ implying no zeros except at $z_0$, contradicts the accumulation point property.

1.12. **Laurent series in an annulus.** Consider an annulus $A = D_R(0) \setminus D_r(0)$ a function $f$ which is holomorphic on $A$. Annuli are, of course, not simply connected.

By the homotopy form of Cauchy’s theorem

$$\int_{\partial D_R(0)} f(z) \, dz = \int_{\partial D_r(0)} f(z) \, dz.$$ 

Applying Cauchy’s formula argument with $f(\zeta)/(\zeta - z)$ we find:

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_R(0)} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{f(\zeta)}{\zeta - z} d\zeta =: f_{in}(z) + f_{out}(z).$$

Now expand the geometric series in each formula

$$f_{in}(z) = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_{\partial D_R(0)} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta \right) z^n$$

power series converges in $D_R(0)$.

Outer series factor out $z$ instead of $\zeta$ in denominator

$$f_{out}(z) = \frac{1}{2\pi i} \frac{1}{z} \int_{\partial D_r(0)} \frac{f(\zeta)}{1 - \frac{\zeta}{z}} d\zeta = \frac{1}{2\pi i} \frac{1}{z} \int_{\partial D_r(0)} f(\zeta) \left( \sum_{n=0}^{\infty} \zeta^n z^{-n} \right) d\zeta$$

power series converges OUTSIDE of $D_r(0)$. 
1.13. **Sequences of holomorphic functions.** Remind about Morera’s theorem.

- Uniform limit of sequence of holomorphic functions is holomorphic (Apply Morera’s theorem).
- Under above assumption the sequence of derivatives also converges uniformly on compact subsets of $\Omega$. Prove that

$$
\sup_K |F'| \leq \frac{1}{d(K, \mathbb{C} \setminus \Omega)} \sup_\Omega |F|
$$

using Cauchy integral formula. Apply this to $F = f_n - f$.

1.14. **Integrals.** If $F(z, s) : \Omega \times [0, 1] \to \mathbb{C}$ is holomorphic in $\Omega$ for each $s \in [0, 1]$ and continuous on $\Omega \times [0, 1]$ then

$$
f(z) = \int_0^1 F(z, s) \, ds
$$

is holomorphic in $\Omega$.

**Proof:** Riemann sums

$$
f_n(z) = \frac{1}{n} \sum_{k=1}^n F(z, \frac{k}{n})
$$

are holomorphic functions in $\Omega$. Want to show that $f_n \to f$ uniformly on any compact subset $K \subset \Omega$. In which case $f$ is holomorphic as well.

Since $K$ is compact $F$ is uniformly continuous on $D \times [0, 1]$ so for any $\varepsilon > 0$ there is $\delta > 0$ so that $|F(z, s) - F(z, t)| \leq \varepsilon$ when $|t - s| \leq \delta$. Take $\frac{1}{n} \leq \delta$ and then

$$
|f_n(z) - f(z)| = \sum |\int_{k/n}^{(k+1)/n} F(z, \frac{k}{n}) - F(z, s)ds| \leq \sum \frac{1}{n} \varepsilon = \varepsilon.
$$

1.15. **Runge’s approximation theorem.** Approximation of general holomorphic functions on compact sets by rational functions / polynomials. Mention relation with Weierstrass polynomial approximation. Note that $1/z$ can’t be approximated by polynomials on the unit circle (would contradict $\int_{\partial D} \frac{dz}{z} = 2\pi i$). Condition will be exactly “no holes” that $K^c$ is connected.

**Theorem:** Any holomorphic function $f$ on a compact set $K$ can be uniformly approximated on $K$ by rational functions with singularities in $K^C$.

If $K^C$ is connected then $f$ can be uniformly approximated by polynomials.

Call $\Omega \supset K$ the open set where $f$ is holomorphic.

**Lemma:** There is a finite collection of line segments $(\gamma_j)_{j=1}^n$ in $\Omega \setminus K$ so that

$$
f(z) = \sum_{j=1}^n \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta.
$$

Let $\rho = \frac{1}{\sqrt{2}} d(K, \Omega^c) > 0$ and let $Q$ be the collection of lattice cubes $\rho(j + [0, 1]^d)$ over $j \in \mathbb{Z}^2$ with side length $\rho$. For any $Q \in Q$ call $\partial Q$ to be
the boundary of $Q$ oriented counter-clockwise. Note by choice of $\rho$ if $Q \in \mathcal{Q}$ intersects $K$ then $Q \subset \Omega$.

Call $\mathcal{Q}_K$ the set of $\rho$ lattice cubes which intersect $K$. If $z \in K$ is not in the union $\cup \partial Q$ then

$$\int_{\partial Q} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \begin{cases} f(z) & \text{if } Q \in \mathcal{Q}_K \\ 0 & \text{else.} \end{cases}$$

Note that $z$ (as above) is in exactly one $Q$ because the interiors of $Q \in \mathcal{Q}$ are disjoint.

Call $\mathcal{F} = \cup_{Q \in \mathcal{Q}_K} Q$. The boundary $\partial \mathcal{F}$ is a finite union of axis parallel line segments $(\gamma_j)_{j=1}^n$.

Notice that

$$\sum_{Q \in \mathcal{Q}_K} \int_{\partial Q} \frac{f(\zeta)}{\zeta - z} \, d\zeta = f(z)$$

but any line segment of a cell boundary $\partial Q$ which is not in $\partial \mathcal{F}$ is integrated over twice in opposite directions so

$$f(z) = \sum_{j=1}^n \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

Lemma: If $\gamma$ is a line segment contained in $\Omega \setminus K$ then $\int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta$ can be approximated uniformly on $z \in K$ by rational functions with singularities on $\gamma$.

Proof: Write

$$g(z) = \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \int_{0}^{1} \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t) \, dt$$

since $d(\gamma(t), K) > 0$ for $t \in [0, 1]$ the integrand $F(z, t)$ above is continuous on $K \times [0, 1]$ and (by compactness) uniformly continuous. Thus the Riemann sums

$$\frac{1}{n} \sum_{k=1}^{n} \frac{f(\gamma(k/n))}{\gamma(k/n) - z} \gamma'(k/n) \to g(z)$$

uniformly on $K$ as $n \to \infty$. Each Riemann sum is a rational function with poles on $\gamma$.

Moving the poles to infinity when $K^c$ is connected:

Lemma: Let $R > 0$ so that $\Omega \subset D_R(0)$. If $z_0 \in \mathbb{C} \setminus D_R(0)$ then $\frac{1}{x-z_0}$ can be uniformly approximated on $K$ by polynomials.

proof:

$$\frac{1}{z - z_0} = -\frac{1}{z_0} \frac{1}{1 - \frac{z}{z_0}} = \sum_{n=0}^{\infty} \frac{z^n}{z_0^{n+1}}$$

which converges uniformly on any compact subset of $D_R(0)$.

Moving a pole slightly:

Lemma: Let $z_0 \in \mathbb{C} \setminus K$ then $\frac{1}{x-z_0}$ can be uniformly approximately on $K$ by rational functions with pole at $z_1$ for any $|z_1 - z_0| < d(z_0, K)/4$. 

proof:
\[
\frac{1}{z - z_0} = \frac{1}{z - z_1 - (z_0 - z_1)} = \frac{1}{z - z_1} \frac{1}{1 - \frac{z_0 - z_1}{z - z_1}} = \sum_{k=0}^{\infty} \frac{(z_0 - z_1)^k}{(z - z_1)^{k+1}}
\]
and note for \( z \in K \)
\[
|z - z_1| \geq |z - z_0| - |z_0 - z_1| \geq d(z_0, K) - d(z_0, K)/4 = 3d(z_0, K)/4 \geq 3|z_0 - z_1|
\]
so
\[
\frac{|z_0 - z_1|}{|z - z_1|} \leq \frac{1}{3} \text{ on } z \in K
\]
and so the geometric series converges uniformly on \( K \).

Lemma: Suppose \( K^c \) is connected and \( z_0 \in K^c \) then \( \frac{1}{z - z_0} \) can be uniformly approximately on \( K \) by polynomials.

Proof: Let \( D_R(0) \supset K \) and \( z_1 \in \mathbb{C} \setminus D_R(0) \). Because \( K^c \) is path connected there is a smooth path \( \gamma \subset K^c \) from \( z_0 \) to \( z_1 \). Since the image of \( \gamma \) is compact there is \( r > 0 \) so that
\[
d(z, K) \geq r \text{ for } z \in \gamma.
\]
Choose a sequence of points \( w_0, \ldots, w_n \in \gamma \) with \( w_0 = z_0 \) and \( w_n = z_1 \) and
\[
|w_{k+1} - w_k| \leq r/4.
\]
Then \( \frac{1}{z - w_k} \) can be uniformly approximated by polynomials in \( \frac{1}{z - w_{k+1}} \) and by induction \( \frac{1}{z - z_0} \) can be approximated by polynomials in \( \frac{1}{z - z_1} \).

Finally \( \frac{1}{z - z_1} \) can be uniformly approximated on \( K \) by polynomials so we are done.

1.16. Schwarz reflection. Symmetry principle: If \( f_+ \) and \( f_- \) are holomorphic in \( \Omega_+ \subset \mathbb{H} \) and \( \Omega_- = \overline{\Omega}_- \) and continuous up to \( I \subset \mathbb{R} \) and agree on \( I \) then \( f \) is holomorphic in \( \Omega = \Omega_+ \cup I \cup \Omega_- \).

Proof: Morera’s theorem divide up triangles crossing the real axis.

Reflection: (For \( f \) defined in \( \Omega_+ \) and taking real values along \( I = \overline{\Omega} \cap \mathbb{R} \)) Check that \( f(\bar{z}) \) is holomorphic by showing power series expansion near every point of \( \Omega_- \).

Note: This is even reflection of real part, and odd reflection of imaginary part.

1.17. Zeros and poles. Zero of order \( m > 0 \) if \( f(z) = (z - z_0)^m g(z) \) and \( g \) holomorphic does not vanish in a neighborhood of \( z_0 \).

Lemma: A nonconstant holomorphic function which vanishes at \( z_0 \in \Omega \) has a zero of finite order.

Pole of order \( m > 0 \) if \( f(z) = (z - z_0)^{-m} g(z) \) where \( g \) holomorphic and does not vanish in a neighborhood of \( z_0 \).

In particular \( 1/f \) is holomorphic near \( z_0 \) and has a zero of order \( m \).

Simple pole/zero if \( m = 1 \).

Lemma: If \( f \) has a pole of order \( m \) at \( z_0 \) then \( f(z) = a_m (z - z_0)^{-m} + \cdots + a_1 (z - z_0)^{-1} + G(z) \) with \( G \) holomorphic near \( z_0 \).
proof: Write \( f(z) = (z - z_0)^{-m}g(z) \) with \( g \) holomorphic and non-zero near \( z_0 \). Write the power series expansion of \( g \) valid in a positive radius neighborhood:

\[
f(z) = (z - z_0)^{-m}g(z) = (z - z_0)^{-m}(A_0 + A_1(z - z_0) + \cdots)
\]

with \( A_0 \neq 0 \).

The negative terms in the Laurent series are called the principal part, the residue of the pole at \( z_0 \) is defined

\[
\text{Res}(f, z_0) = a_{-1}
\]

The importance of the residue is that \((z - z_0)^{-m}\) has a primitive in \( \mathbb{C} \setminus \{0\}\) for any \( m > 1 \) so for any small circle centered at \( z_0 \)

\[
\int_{C_r(z_0)} f(z) \, dz = \int_{C_r(z_0)} P(z) \, dz = \int_{C_r(z_0)} \text{Res}(f, z_0) \frac{1}{z - z_0} \, dz = 2\pi i \text{Res}(f, z_0).
\]

where \( P(z) \) is the principal part of the Laurent series of \( f \) at \( z_0 \).

If \( f \) has a pole of order \( m \) at \( z_0 \) then

\[
\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \left( \frac{d}{dz} \right)^{m-1} (z - z_0)^m f(z).
\]

(Note case of simple pole is particularly simple.

1.18. Residue formula. Suppose \( f \) holomorphic in an open set containing a closed disc \( D \) with boundary circle \( C \) except for a pole at \( z_0 \in D \) then

\[
\int_C f(z) \, dz = 2\pi i \text{Res}(f, z_0).
\]

Use keyhole contour and previous set up with principal part of the Laurent expansion.

Generalize to (1) multiple poles, (2) general “toy contours”.

Example computations

\[
\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \pi
\]

Example

\[
\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \, dx = \frac{\pi}{\sin(\pi a)}
\]

(contour is a rectangle boundary of \([-R, R] \times [0, 2\pi]\)) pole of \( f(z) = \frac{e^{az}}{1 + e^z} \) at \( z = i\pi \).

1.19. Singularities. A function \( f \) is said to have an isolated singularity at \( z_0 \) if \( f \) is defined and holomorphic in a punctured neighborhood \( D_r(z_0) \setminus \{z_0\} \).

An isolated singularity is said to be removable if \( f \) can be extended to be holomorphic in a neighborhood of \( z_0 \).

Theorem: (Riemann removable singularity theorem) If \( f \) has an isolated singularity at \( z_0 \) and \( f \) is bounded in a punctured neighborhood of \( z_0 \) then the singularity is removable.
proof: Integrate $f(\zeta)/(\zeta - z)$ on a keyhole contour excising $z_0$ and $z \neq z_0$ to show

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta = G(z).$$

The function $G(z)$ is holomorphic on $D_r(z)$ and extends $f$.

Corollary: If $f$ has an isolated singularity at $z_0$ then $f$ has a pole at $z_0$ if and only if $|f(z)| \to \infty$ as $z \to z_0$.

proof: If $f(z)$ has a pole at $z_0$ then $1/f$ has a zero at $z_0$ so $|f(z)| \to \infty$ as $z \to z_0$. If $|f(z)| \to \infty$ as $z \to z_0$ then $1/f$ is bounded near $z_0$ and hence has a removable singularity at $z_0$, since $|1/f| \to 0$ as $z \to z_0$ the only continuous extension of $g = 1/f$ at $z_0$ is by $g(z_0) = 0$. Thus $f$ has a pole at $z_0$.

Types of isolated singularities

- Removable ($f$ bounded near $z_0$)
- Pole
- Essential singularity

The function $e^{1/z}$ has an essential singularity at $z = 0$, note that as $z$ approaches zero along $\mathbb{R}$ left limit is 0 right limit is $\infty$, along the imaginary axis limits are bounded.

Theorem: (Casorati-Weierstrass) If $f$ has an essential singularity at $z_0$ then the image of $D_r(z_0) \setminus \{z_0\}$ is dense in $\mathbb{C}$ for all $r > 0$.

Proof: Suppose otherwise, if a $\delta$ neighborhood of $w$ is missed consider $g(z) = 1/(f(z) - w)$ which is bounded and so has a removable singularity at $z_0$. Then $f(z) - w$ is either bounded or has a pole at $z_0$ (in the case $g(z_0) = 0$).

Stronger results exist (Big Picard): Under the same hypothesis $f$ attains every value in $\mathbb{C}$ infinitely many times with at most one exception. This will come later in the class.

1.20. Meromorphic functions. A function $f$ is called meromorphic on a domain $\Omega$ in the complex plane if there is a sequence of points $\{z_1, \ldots, z_n\}$ with no limit points in $\Omega$ and such that $f$ is holomorphic on $\Omega \setminus Z$ and $f$ has poles at the points of $Z$.

We can also discuss meromorphic functions on the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. Neighborhoods of $\infty$ in the extended plane are complements of closed discs $\{|z - z_0| < r\}$. If $f$ is holomorphic in a (punctured) neighborhood of $\infty$ and the function $F(z) = f(1/z)$ has a removable singularity at zero then we say that $f$ is holomorphic at infinity. If $F$ has a pole at 0 then we say $f$ has a pole at infinity, and same for essential singularity.

By this means we can define a meromorphic function on $\mathbb{C}^*$:

Lemma: Any meromorphic function on $\mathbb{C}^*$ is a rational function.

proof: $f(1/z)$ has either a pole or a removable singularity at $0$ and hence is holomorphic in a punctured neighborhood of $0$, i.e. $f$ is holomorphic in a neighborhood of $\infty$ meaning $f$ can only have at most finitely many poles in the extended plane $z_1, \ldots, z_n$. 
We will subtract the principal part at each pole and then apply Liouville’s theorem to what is left. Near each pole $z_k$ we can write

$$f(z) = P_k(z) + g_k(z)$$

where $g_k$ is holomorphic near $z_k$ and $P_k(z)$ is a polynomial in $1/(z - z_k)$, in particular a rational function. Similarly we can write

$$f(1/z) = P_\infty(z) + g_\infty(z)$$

where $P_\infty$ is a polynomial in $1/z$ and $g$ is holomorphic at $0$. Then define

$$H(z) = f(z) - P_\infty(1/z) - \sum_{k=1}^n P_k(z)$$

where $P_\infty(1/z)$ is a polynomial and $P_k(z)$ are polynomials in $1/(z - z_k)$. Now $H$ is entire and $H(1/z)$ is also bounded in a punctured neighborhood of the origin. In particular $H$ is entire and bounded and hence constant by Liouville.

Note one consequence: rational functions are determined by their principal parts at their poles. Rational functions with poles at $\{z_1, \ldots, z_n\}$ (possibly repeated, possibly $\infty$) is a vector space of dimension $n + 1$.

1.21. **Riemann sphere.** Very brief intro will think more about it later. The extended plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ can be thought of as a sphere. Show the stereographic projection.

1.22. **Argument principle.** Eventually we want to talk about

$$\log f(z) = \log |f(z)| + i\arg(f(z))$$

the argument function is multi-valued so we will need to treat this carefully.

However the logarithmic derivative is a well defined meromorphic function

$$\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)}$$

(This must be the derivative of any branch of $\log f(z)$ based on the identity $e^{\log f(z)} = z$) which has poles at the zeros of $f$.

The contour integral on a curve $\gamma$

$$\int_{\gamma} \frac{f'(z)}{f(z)} \, dz$$

can be interpreted as the change of the argument along the curve $\gamma$.

If $f$ has a zero at $z_0$ of order $m$ then

$$f(z) = (z - z_0)^m g(z)$$

and

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}$$

where $g'/g$ is holomorphic near $z_0$ since $g$ is nonzero there.
Similarly at a pole $f(z) = (z - z_0)^{-m}h(z)$

$$\frac{f'(z)}{f(z)} = \frac{-m}{z - z_0} + \frac{h'(z)}{h(z)}.$$ 

Thus the residue of $f'/f$ at a zero or pole of $f$ is

$$\text{Res}(f, z_0) = m$$

where $m \in \mathbb{Z}$ is the order of the zero / negative order of the pole.