COURSE OUTLINE MATH 6220

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1. Basics

Stein and Shakarchi Ch 1

1.1. **Complex plane.**
Real and Imaginary parts, complex addition, multiplication, geometry, triangle inequality, complex conjugate, $\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$ etc, complex magnitude, $1/z = \bar{z}/|z|^2$, polar form $z = re^{i\theta}$ where $r = |z|$, $\theta = \arg(z)$, and Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, complex multiplication as homothety.

1.2. **Topology of $\mathbb{C}$.**
Notion of convergence, Cauchy sequences, $\mathbb{C}$ is complete, notation for discs $D_r(z_0)$, circles $C_r(z_0)$, unit disk $\mathbb{D}$, unit circle $\partial\mathbb{D}$, open, closed, bounded, compact (closed and bounded iff sequentially compact iff open cover compact ... nested intersection property may be useful later), connected set, equivalence of connected and path connected for open sets in $\mathbb{C}$.

1.3. **Holomorphic functions.**
Continuity (same as for functions on $\mathbb{R}^2$), Holomorphic at z_0 if the difference quotients converge, limit is called the derivative, function is holomorphic on Ω open if it is complex differentiable at every point, on C closed if there is Ω open containing C on which it is holomorphic, on all of $\mathbb{C}$ entire. $f(z) = z$ and any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$. $f(z)$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and $f'(z) = -1/z^2$. $f(z) = \bar{z}$ is NOT holomorphic. Useful also to write differentiability as

$$f(z_0 + h) - f(z_0) - ah = h\psi(h)$$

where ψ is defined for small enough $|h|$ and $\psi(h) \to 0$ as $h \to 0$. Also can write this as $= o(h)$ (with this as the precise defn). Proposition: $(f+g)' = f' + g'$, $(fg)' = f'g + g'f$, if $g(z_0) \neq 0$ then $(f/g)' = (f'g - g'f)/g^2$, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic then $(g \circ f)'(z) = g'(f(z))f'(z)$ for $z \in \Omega$.

In particular polynomials are holomorphic and complex derivatives have the known formulae.

1.4. **Cauchy-Riemann equations.**
Compute $f'(z)$ two ways with $h = h_1 \in \mathbb{R}$ and $h = ih_2$ with $h_2 \in \mathbb{R}$, find $\partial_x f = \frac{1}{i}\partial_y f$ and

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_y u = -\partial_x v.$$

Also note that $\partial_x^2 u = \partial_y(-\partial_y u) = -\partial_y^2 u$ so u is *harmonic* (assuming sufficient regularity for now).

Define the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) \ \text{ and } \ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$$

Then if f is holomorphic then

$$\frac{\partial f}{\partial \bar{z}} = 0 \ \text{ and } \ f'(z) = \frac{\partial f}{\partial z} = 2\frac{\partial u}{\partial z}$$

also jacobian formula for $F(x,y) = (u(z), v(z))$

$$\det(DF(x,y)) = |f'(z)|^2.$$

Note that sign is always positive so Df is orientation preserving, anti-holomorphic functions reverse orientation.

$$\det(DF) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = |f'(z)|^2$$

CONVERSE: If u and v are C^1 and satisfy CR equations then $f(z) = u + iv$ is holomorphic.

1.5. **Power series.** $e^z = \sum z^n/n!$, note by triangle ineq $|e^z| \leq e^{|z|}$ which implies absolute summability for all $z \in \mathbb{C}$. Geometric series $\sum^N z^n = \frac{1-z^{N+1}}{1-z}$ converges in $|z| < 1$ to the function $1/(1-z)$ holomorphic in $\mathbb{C} \setminus \{1\}$.

Note that if power series converges absolutely at z_0 then it also converges absolutely for all z with $|z| \leq |z_0|$.

Theorem: Given series $\sum a_n z^n$ there is $0 \leq R \leq \infty$ called radius of convergence so that for $|z| < R$ power series converges absolutely, for $|z| > R$ diverges, and formula

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

Proof: Call $L = 1/R$ and let $\varepsilon > 0$ so that $(L+\varepsilon)|z| = r < 1$ then

$$|a_n||z|^n = (|a_n|^{1/n}|z|)^n \leq ((L+\varepsilon)|z|)^n = r^n$$

for n sufficiently large. Since this geometric series is absolutely summable we get convergence.

If $|z| > R$ similar argument shows that there is a subsequence of terms so that $|a_n||z|^n \to \infty$ which means that the partial sums cannot be Cauchy and hence cannot converge.

Trigonometric functions

$$\cos z = \sum (-1)^n z^{2n}/(2n)! \ \text{ and } \ \sin z = \sum (-1)^n z^{2n+1}/(2n+1)!$$

and

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \ \text{ and } \ \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

also reverse formulae $e^{iz} = \cos z + i \sin z$ and e^{-iz}.

Note for a function given by a power series $\overline{f(z)} = f(\bar{z})$.

Differentiating term by term: Note if $f_n(z) \to f(z)$ uniformly this does not generally imply that the derivatives $f_n'(z) \to f(z)$ (at least in real variable problems!). For power series it is true inside the radius of convergence. If we establish that $\frac{f_n(z)-f_n(z_0)}{z-z_0}$ converges to $\frac{f(z)-f(z_0)}{z-z_0}$ *uniformly* then the result is true.

Apply to power series using $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$

$$|\sum_N^\infty a_n \frac{(z_0+h)^n - z_0^n}{h}| =\leq \sum_N^\infty n|a_n||z_0|^{n-1}$$

this is $\frac{n}{|z_0|}|a_n||z_0|^n$ but the coefficient $b_n = \frac{n}{|z_0|}|a_n|$ has the same radius of convergence since

$$\limsup b_n^{1/n} = \limsup n^{1/n}|z_0|^{-1/n}|a_n|^{1/n} = 1/R.$$

Corollary: Power series are infinitely differentiable, derivatives are the termwise derivative power series, and the radii of convergence of all derivatives are the same.

Can also center power series at z_0. A function is called *analytic* at z_0 if it has a power series representation at z_0 in a positive radius neighborhood. It turns out that holomorphic and analytic on an open domain Ω are equivalent!

1.6. **Integration on curves.** It is useful to distinguish between the one-dimensional objects with orientation in $\mathbb{C}$ and their parametrizations.

Parametrized curve is a map $z(t) : [a,b] \to \mathbb{C}$, called smooth if z is C^1 on $[a,b]$ and $z'(t) \neq 0$ (one-sided difference quotients at a, b). Recall issue with $z' = 0$. Called piecewise smooth if z is continuous and $[a,b]$ can be partitioned into (almost) disjoint closed subintervals on which z is smooth.

Two parametrized curves z on $[a,b]$ and w on $[c,d]$ are equivalent if there is a strictly monotone increasing and smooth change of variables $t : [c,d] \to [a,b]$ and

$$w(s) = z(t(s)).$$

The condition $t'(s) > 0$ guarantees that z and w have the same orientation.

An equivalent family of smooth parametrized curves is an *(oriented) curve* γ in $\mathbb{C}$. Piece-wise smooth curves are defined similarly. γ^- is the same curve with reverse orientation, can also write $-\gamma$.

Endpoints of curve are $z(a)$ and $z(b)$ (independent of parametrization). Curve is *closed* if endpoints are the same. Curve is *simple* if no self-intersections.

For example $C_r(z_0)$ is an un-oriented curve a positive (counter-clockwise) parametrization is

$$z(t) = z_0 + re^{it} \; t \in [0, 2\pi]$$

negative (clockwise) oriented parametrization

$$z(t) = z_0 + re^{-it} \; t \in [0, 2\pi].$$

The integral on a piece-wise smooth curve is

$$\int_\gamma f(z)\ dz = \int_a^b f(z(t))z'(t)dt$$

where we need to justify that the definition does not depend on choice of parametrization: with $w(s) = z(t(s))$ as before $dt = t'(s)ds$ so change of variables formula gives

$$\int_a^b f(z(t))z'(t)dt = \int_c^d f(w(s))z'(t(s))t'(s)ds = \int_c^d f(w(s))w'(s)\ ds.$$

The length of a piecewise smooth curve is defined

$$\text{length}(\gamma) = \int_a^b |z'(t)|\ dt = \sup_P \sum_{i=0}^N |z(t_{i+1}) - z(t_i)|.$$

Path length integral

$$\int_\gamma f(z)d|z| = \int_a^b f(z(t))|z'(t)|dt$$

Path integration is linear, changes sign under orientation reversal, natural bound

$$|\int_\gamma f(z)\ dz| \leq (\sup_\gamma |f|)\text{length}(\gamma).$$

More generally

$$|\int_\gamma f(z)\ dz| \leq \int_\gamma |f(z)|\ d|z|$$

1.7. **Primitives.** A function f on a domain U has a *primitive* F if F is holomorphic on U and $F'(z) = f(z)$.

Lemma: If F is a primitive for f continuous then

$$\int_\gamma f(z)\ dz = F(w_2) - F(w_1)$$

for any curve γ beginning at w_2 and ending at w_1.

In particular if f has a primitive on U then

$$\int_\gamma f\ dz = 0$$

for any closed curve γ in U.

Note: NOT every holomorphic function on a domain U has a primitive on U, e.g. $1/z$ on $\mathbb{D} \setminus \{0\}$.

1.8. **Goursat's theorem.** If Ω is an open set in $\mathbb{C}$, f is holomorphic in Ω and T is a triangle whose interior is also contained in Ω then

$$\int_T f(z)\ dz = 0$$

We use the following property of compact sets: if $K_1 \supset K_2 \supset \cdots$ is a nested collection of compact sets then $\cap K_j$ is nonempty. If the diameter $\operatorname{diam}(K_j) \to 0$ then the intersection is a singleton.

Remark: the proof technique also works for rectangles but it is easier to derive the rectangle case from the triangle case than vice versa.

Derive the same result for rectangles by splitting into two triangles.

Idea: reduce to zooming in, holomorphic functions look locally linear for zoomed in case, linear functions have a primitive and satisfy the theorem.

Why is it **not** true for C^1 functions.

1.9. **Local existence of primitives.** A holomorphic function on an open disc has a primitive on that disc.

Proof: Define primitive by integrating from 0 to z along right angle path γ_z. Use Goursat to write $F(z+h) - F(z) = \int_\eta f(z)\ dz$ where η is the line segment from z to $z+h$. Use f continuous write $f(w) = f(z) + \psi(w)$ and compute.

Corollary (Cauchy's theorem) The integral of a holomorphic function in a disc on any closed curve in that disc is zero.

1.10. **Homotopy.** Two parametrized curves in a set U are **homotopic** if there is a map $\psi(t,s) : [a,b] \times [0,1] \to U$ continuous in s and piecewise smooth in t for each s so that

$$\psi(t,0) = \gamma \ \text{ and } \ \psi(t,1) = \eta.$$

Describe as continuous deformation of curves, draw pictures (1) two paths, (2) circle around missing point.

For non-closed paths we will include in the definition that homotopy fixes end points. For closed paths we will include in the definition that homotopy preserves the property of being closed.

Definition generalizes to oriented curves, creates an equivalence relation on curves in U.

Homotopy form of Cauchy's theorem: if f holomorphic in U and γ and η are homotopic in U then

$$\int_\gamma f(z)dz = \int_\eta f(z)\ dz$$

I.e. contour integral is homotopy invariant, constant on homotopy equivalence classes. Proof: 1. do case of nearby curves, 2. reduce to case of nearby curves by the homotopy

Note: If γ is a closed curve homotopic to a point then $\int_\gamma f(z)\ dz = 0$.

Define U is simply connected if it is connected and every closed curve is homotopic to a point. Non-examples: annulus, set minus several points.

If U is simply connected then $\int_\gamma f(z)dz = 0$ for every closed curve γ contained in U.

Examples: Convex implies simply connected.

Holomorphic functions on simply connected domains have a primitive: define primitive by integrating on arbitrary path from fixed base point, use Cauchy's theorem to show the definition is good.

1.11. **Cauchy's integral formula.** Statement f holomorphic in $\overline{D(z_0, r)}$ then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

-Note $f(\zeta)/(\zeta - z)$ not holomorphic on the inside of disc so no contradiction.

-Keyhole contour with parameters $\varepsilon > 0$ and $\delta > 0$. Fix $\varepsilon > 0$ and send $\delta \to 0$ to show equality of outer and inner integrals.

-Write

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta) - f(z)}{\zeta - z} + \frac{f(z)}{\zeta - z}$$

and use holomorphicity so first term converges to $f'(z)$ as $\zeta \to z$.

Holomorphic implies infinitely many complex derivatives and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(\zeta - z)^{n+1}} \, d\zeta.$$

Proof: If f holomorphic in a neighborhood of z_0 can choose a circle $C_r(z_0)$ so that f is holomorphic in neighborhood of C. For z in that disc take Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta = f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)} \, d\zeta = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \, d\zeta$$

and expand the geometric series.

COROLLARY OF PROOF: If f is holomorphic in $D_r(z_0)$ then the power series expansion for f centered at z_0 has radius of convergence at least r.

In particular *holomorphic in an open set U is equivalent to analytic in U.*

COROLLARY: If f has a primitive in a disc $D_r(z_0)$ then f is holomorphic in $D_r(z_0)$

COROLLARY: (Morera's theorem) If $\int_T f(z) \, dz = 0$ for all triangles T in a disc $D_r(z_0)$ then f has a primitive and is holomorphic in $D_r(z_0)$.

Cauchy inequalities

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi R^n} \sup_{D_r(z)} |f|$$

Liouville Theorem- bounded entire imples constant by Cauchy inequality for $n = 1$.

Every non-constant polynomial has a root in $\mathbb{C}$: if not $1/P$ would be a bounded holomorphic function (since polynomials go to ∞ at ∞).

Every degree $n \geq 1$ polynomial has exactly n roots in $\mathbb{C}$ and $P(z) = a_n(z-z_1)\cdots(z-z_n)$.

proof: P has a root z_1 then consider expanding each term by binomial formula

$$P(z) = P(z - z_1 + z_1) = b_n(z-z_1)^n + b_{n-1}(z-z_1)^n + \cdots + b_1(z-z_1) + b_0$$

with $b_n = a_n$, must be $b_0 = 0$ since $P(z_1) = b_0 = 0$. Thus $P(z) = (z - z_1)Q(z)$ where Q has degree $n-1$. Induction.

If f holomorphic on connected Ω vanishes on a set E with an accumulation point then f is zero. (More generally, unique continuation, f is determined by its values on any set with an accumulation point, just think about $f-g$ for any two functions with the same values on E).

proof: 1. Show that if $z_0 = 0$ is an accumulation point of $\{f = 0\}$ then $f = 0$ in a neighborhood of z_0. 2. Consider $U = \{f = 0\}^o$ this is open by definition, but also closed by the part 1, thus U is open and closed in Ω and f is constant in Ω (connected).

Proof of 1: Power series expansion in $D_r(z_0)$, if all coefficients are zero done. Otherwise

$$f(z) = a_m(z-z_0)^m(1+g(z)) \quad \text{with} \quad a_m \neq 0$$

where $g(z) \to 0$ as $z \to 0$. But then $|f(z)| \geq |a_m|/2|z-z_0|^m$ on sufficiently small disc around z_0 implying no zeros except at z_0, contradicts the accumulation point property.

1.12. **Laurent series in an annulus.** Consider an annulus $A = D_R(0) \setminus D_r(0)$ a function f which is holomorphic on $\overline{A}$. Annuli are, of course, not simply connected.

By the homotopy form of Cauchy's theorem

$$\int_{\partial D_R(0)} f(z)\ dz = \int_{\partial D_r(0)} f(z)\ dz.$$

Applying Cauchy's formula argument with $f(\zeta)/(\zeta - z)$ we find:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D_R(0)} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i}\int_{\partial D_r(0)} \frac{f(\zeta)}{\zeta - z} d\zeta =: f_{in}(z) + f_{out}(z).$$

Now expand the geometric series in each formula

$$f_{in}(z) = \sum_{n=0}^{\infty} [\frac{1}{2\pi i}\int_{\partial D_R(0)} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta] z^n$$

power series converges in $D_R(0)$.

Outer series factor out z instead of ζ in denominator

$$f_{out}(z) = \frac{1}{2\pi i}\frac{1}{z}\int_{\partial D_r(0)} \frac{f(\zeta)}{1 - \frac{\zeta}{z}} d\zeta = \frac{1}{2\pi i}\frac{1}{z}\int_{\partial D_r(0)} f(\zeta)(\sum_{n=0}^{\infty} \zeta^n z^{-n}) d\zeta$$

power series converges OUTSIDE of $\overline{D_r(0)}$.

1.13. **Sequences of holomorphic functions.** Remind about Morera's theorem.

-Uniform limit of sequence of holomorphic functions is holomorphic (Apply Morera's theorem).

-Under above assumption the sequence of derivatives also converges uniformly on compact subsets of Ω. Prove that

$$\sup_K |F'| \leq \frac{1}{d(K, \mathbb{C} \setminus \Omega)} \sup_\Omega |F|$$

using Cauchy integral formula. Apply this to $F = f_n - f$.

1.14. **Integrals.** If $F(z,s) : \Omega \times [0,1] \to \mathbb{C}$ is holomorphic in Ω for each $s \in [0,1]$ and continuous on $\Omega \times [0,1]$ then

$$f(z) = \int_0^1 F(z,s)\ ds$$

is holomorphic in Ω.

Proof: Riemann sums

$$f_n(z) = \frac{1}{n} \sum_{k=1}^n F(z, \frac{k}{n})$$

are holomorphic functions in Ω. Want to show that $f_n \to f$ uniformly on any compact subset $K \subset \Omega$. In which case f is holomorphic as well.

Since K is compact F is uniformly continuous on $D \times [0,1]$ so for any $\varepsilon > 0$ there is $\delta > 0$ so that $|F(z,s) - F(z,t)| \leq \varepsilon$ when $|t - s| \leq \delta$. Take $\frac{1}{n} \leq \delta$ and then

$$|f_n(z) - f(z)| = \sum | \int_{k/n}^{(k+1)/n} F(z, \frac{k}{n}) - F(z,s) ds| \leq \sum \frac{1}{n}\varepsilon = \varepsilon.$$

1.15. **Runge's approximation theorem.** Approximation of general holomorphic functions on compact sets by rational functions / polynomials. Mention relation with Weierstrass polynomial approximation. Note that $1/z$ can't be approximated by polynomials on the unit circle (would contradict $\int_{\partial \mathbb{D}} \frac{dz}{z} = 2\pi i$). Condition will be exactly "no holes" that K^c is connected.

Theorem: Any holomorphic function f on a compact set K can be uniformly approximated on K by rational functions with singularities in K^C. If K^C is connected then f can be uniformly approximated by polynomials.

Call $\Omega \supset\supset K$ the open set where f is holomorphic.

Lemma: There is a finite collection of line segments $(\gamma_j)_{j=1}^n$ in $\Omega \setminus K$ so that

$$f(z) = \sum_{j=1}^n \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z}\ d\zeta.$$

Let $\rho = \frac{1}{\sqrt{2}} d(K, \Omega^c) > 0$ and let $\mathcal{Q}$ be the collection of lattice cubes $\rho(j + [0,1]^d)$ over $j \in \mathbb{Z}^2$ with side length ρ. For any $Q \in \mathcal{Q}$ call ∂Q to be the boundary of Q oriented counter-clockwise. Note by choice of ρ if $Q \in \mathcal{Q}$ intersects K then $Q \subset \Omega$.

Call $\mathcal{Q}_K$ the set of ρ lattice cubes which intersect K. If $z \in K$ is not in the union $\cup \partial Q$ then

$$\int_{\partial Q} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \begin{cases} f(z) & z \in Q \\ 0 & \text{else.} \end{cases} \quad \text{for} \quad Q \in \mathcal{Q}_K$$

Note that z (as above) is in exactly one Q because the interiors of $Q \in \mathcal{Q}$ are disjoint.

Call $F = \cup_{Q \in \mathcal{Q}_K} Q$. The boundary ∂F is a finite union of axis parallel line segments $(\gamma_j)_{j=1}^n$.

Notice that

$$\sum_{Q \in \mathcal{Q}_K} \int_{\partial Q} \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z)$$

but any line segment of a cell boundary ∂Q which is not in ∂F is integrated over twice in opposite directions so

$$f(z) = \sum_{j=1}^{n} \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Lemma: If γ is a line segment contained in $\Omega \setminus K$ then $\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$ can be approximated uniformly on $z \in K$ by rational functions with singularities on γ.

Proof: Write

$$g(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t) dt$$

since $d(\gamma(t), K) > 0$ for $t \in [0, 1]$ the integrand $F(z, t)$ above is continuous on $K \times [0, 1]$ and (by compactness) uniformly continuous. Thus the Riemann sums

$$\frac{1}{n} \sum_{k=1}^{n} \frac{f(\gamma(k/n))}{\gamma(k/n) - z} \gamma'(k/n) \to g(z)$$

uniformly on K as $n \to \infty$. Each Riemann sum is a rational function with poles on γ.

Moving the poles to infinity when K^c is connected:

Lemma: Let $R > 0$ so that $\Omega \subset D_R(0)$. If $z_0 \in \mathbb{C} \setminus D_R(0)$ then $\frac{1}{z - z_0}$ can be uniformly approximated on K by polynomials.

proof:

$$\frac{1}{z - z_0} = -\frac{1}{z_0} \frac{1}{1 - \frac{z}{z_0}} = \sum_{n=0}^{\infty} -\frac{z^n}{z_0^{n+1}}$$

which converges uniformly on any compact subset of $D_R(0)$.

Moving a pole slightly:

Lemma: Let $z_0 \in \mathbb{C} \setminus K$ then $\frac{1}{z - z_0}$ can be uniformly approximately on K by rational functions with pole at z_1 for any $|z_1 - z_0| < d(z_0, K)/4$.

proof:

$$\frac{1}{z-z_0} = \frac{1}{z-z_1-(z_0-z_1)} = \frac{1}{z-z_1}\frac{1}{1-\frac{z_0-z_1}{z-z_1}} = \sum_{k=0}^{\infty}\frac{(z_0-z_1)^k}{(z-z_1)^{k+1}}$$

and note for $z \in K$

$$|z-z_1| \geq |z-z_0|-|z_0-z_1| \geq d(z_0,K)-d(z_0,K)/4 = 3d(z_0,K)/4 \geq 3|z_0-z_1|$$

so

$$\frac{|z_0-z_1|}{|z-z_1|} \leq \frac{1}{3} \quad \text{on} \quad z \in K$$

and so the geometric series converges uniformly on K.

Lemma: Suppose K^c is connected and $z_0 \in K^c$ then $\frac{1}{z-z_0}$ can be uniformly approximately on K by polynomials.

proof: Let $D_R(0) \supset K$ and $z_1 \in \mathbb{C} \setminus D_R(0)$. Because K^c is path connected there is a smooth path $\gamma \subset K^c$ from z_0 to z_1. Since the image of γ is compact there is $r > 0$ so that

$$d(z,K) \geq r \quad \text{for} \quad z \in \gamma.$$

Choose a sequence of points $w_0, \dots, w_n \in \gamma$ with $w_0 = z_0$ and $w_n = z_1$ and $|w_{k+1} - w_k| \leq r/4$. Then $\frac{1}{z-w_k}$ can be uniformly approximated by polynomials in $\frac{1}{z-w_{k+1}}$ and by induction $\frac{1}{z-z_0}$ can be approximated by polynomials in $\frac{1}{z-z_1}$.

Finally $\frac{1}{z-z_1}$ can be uniformly approximated on K by polynomials so we are done.

1.16. **Schwarz reflection.** Symmetry principle: If f_+ and f_- are holomorphic in $\Omega_+ \subset \mathbb{H}$ and $\Omega_- = \bar{\Omega}_-$ and continuous up to $I \subset \mathbb{R}$ and agree on I then f is holomorphicin $\Omega = \Omega_+ \cup I \cup \Omega_-$.

Proof: Morera's theorem divide up triangles crossing the real axis.

Reflection: (For f defined in Ω_+ and taking real values along $I = \overline{\Omega} \cap \mathbb{R}$) Check that $\overline{f(\bar{z})}$ is holomorphic by showing power series expansion near every point of Ω_-.

Note: This is even reflection of real part, and odd reflection of imaginary part.

1.17. **Zeros and poles.** Zero of order $m > 0$ if $f(z) = (z-z_0)^m g(z)$ and g holomorphic does not vanish in a neighborhood of z_0.

Lemma: A nonconstant holomorphic function which vanishes at $z_0 \in \Omega$ has a zero of finite order.

Pole of order $m > 0$ if $f(z) = (z-z_0)^{-m} g(z)$ where g holomorphic and does not vanish in a neighborhood of z_0.

In particular $1/f$ is holomorphic near z_0 and has a zero of order m.

Simple pole/zero if $m = 1$.

Lemma: If f has a pole of order m at z_0 then $f(z) = a_{-m}(z-z_0)^{-m} + \cdots + a_{-1}(z-z_0)^{-1} + G(z)$ with G holomorphic near z_0.

proof: Write $f(z) = (z - z_0)^{-m} g(z)$ with g holomorphic and non-zero near z_0. Write the power series expansion of g valid in a positive radius neighborhood:

$$f(z) = (z - z_0)^{-m} g(z) = (z - z_0)^{-m}(A_0 + A_1(z - z_0) + \cdots)$$

with $A_0 \neq 0$.

The negative terms in the **Laurent series** are called the **principal part**, the **residue** of the pole at z_0 is defined

$$\mathrm{Res}(f, z_0) = a_{-1}$$

The importance of the residue is that $(z - z_0)^{-m}$ has a primitive in $\mathbb{C} \setminus \{0\}$ for any $m > 1$ so for any small circle centered at z_0

$$\int_{C_r(z_0)} f(z)\, dz = \int_{C_r(z_0)} P(z)\, dz = \int_{C_r(z_0)} \mathrm{Res}(f, z_0) \frac{1}{z - z_0}\, dz = 2\pi i \mathrm{Res}(f, z_0).$$

where $P(z)$ is the principal part of the Laurent series of f at z_0.

If f has a pole of order m at z_0 then

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \left(\frac{d}{dz}\right)^{m-1} (z - z_0)^m f(z).$$

(Note case of simple pole is particularly simple.

1.18. **Residue formula.** Suppose f holomorphic in an open set containing a closed disc $\overline{D}$ with boundary circle C except for a pole at $z_0 \in D$ then

$$\int_C f(z)\, dz = 2\pi i \mathrm{Res}(f, z_0).$$

Use keyhole contour and previous set up with principal part of the Laurent expansion.

Generalize to (1) multiple poles, (2) general “toy contours”.

Example computations

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\, dx = \pi$$

Example

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x}\, dx = \frac{\pi}{\sin(\pi a)}$$

(contour is a rectangle boundary of $[-R, R] \times [0, 2\pi]$) pole of $f(z) = \frac{e^{az}}{1+e^z}$ at $z = i\pi$.

1.19. **Singularities.** A function f is said to have an isolated singularity at z_0 if f is defined and holomorphic in a punctured neighborhood $D_r(z_0) \setminus \{z_0\}$. An isolated singularity is said to be *removable* if f can be extended to be holomorphic in a neighborhood of z_0.

Theorem: (Riemann removable singularity theorem) If f has an isolated singularity at z_0 and f is bounded in a punctured neighborhood of z_0 then the singularity is removable.

proof: Integrate $f(\zeta)/(\zeta - z)$ on a keyhole contour excising z_0 and $z \neq z_0$ to show

$$f(z) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta = G(z).$$

The function $G(z)$ is holomorphic on $D_r(z)$ and extends f.

Corollary: If f has an isolated singularity at z_0 then f has a pole at z_0 if and only if $|f(z)| \to \infty$ as $z \to z_0$.

proof: If $f(z)$ has a pole at z_0 then $1/f$ has a zero at z_0 so $|f(z)| \to \infty$ as $z \to z_0$. If $|f(z)| \to \infty$ as $z \to z_0$ then $1/f$ is bounded near z_0 and hence has a removable singularity at z_0, since $|1/f| \to 0$ as $z \to z_0$ the only continuous extension of $g = 1/f$ at z_0 is by $g(z_0) = 0$. Thus f has a pole at z_0.

Types of isolated singularities

- Removable (f bounded near z_0)
- Pole
- Essential singularity

The function $e^{1/z}$ has an essential singularity at $z = 0$, note that as z approaches zero along $\mathbb{R}$ left limit is 0 right limit is ∞, along the imaginary axis limits are bounded.

Theorem: (Caseroti-Weierstrass) If f has an essential singularity at z_0 then the image of $D_r(z_0) \setminus \{z_0\}$ is dense in $\mathbb{C}$ for all $r > 0$.

Proof: Suppose otherwise, if a δ neighborhood of w is missed consider $g(z) = 1/(f(z) - w)$ which is bounded and so has a removable singularity at z_0. Then $f(z) - w$ is either bounded or has a pole at z_0 (in the case $g(z_0) = 0$).

Stronger results exist (Big Picard): Under the same hypothesis f attains every value in $\mathbb{C}$ infinitely many times with at most one exception. This will come later in the class.

1.20. **Meromorphic functions.** A function f is called *meromorphic* on a domain Ω in the complex plane if there is a sequence of points $\{z_1, \ldots, z_n\}$ with no limit points in Ω and such that f is holomorphic on $\Omega \setminus Z$ and f has poles at the points of Z.

We can also discuss meromorphic functions on the *extended complex plane* $\mathbb{C}* = \mathbb{C} \cup \{\infty\}$. Neighborhoods of ∞ in the extended plane are complements of closed discs $\{|z - z_0| > r\}$. If f is holomorphic in a (punctured) neighborhood of ∞ and the function $F(z) = f(1/z)$ has a removable singularity at zero then we say that f is holomorphic at infinity. If F has a pole at 0 then we say f has a pole at infinity, and same for essential singularity.

By this means we can define a meromorphic function on $\mathbb{C}^*$:

Lemma: Any meromorphic function on $\mathbb{C}^*$ is a rational function.

proof: $f(1/z)$ has either a pole or a removable singularity at 0 and hence is holomorphic in a punctured neighborhood of 0, i.e. f is holomorphic in a neighborhood of ∞ meaning f can only have at most finitely many poles in the extended plane $z_1, \ldots, z_n$.

We will subtract the principal part at each pole and then apply Liouville's theorem to what is left. Near each pole z_k we can write

$$f(z) = P_k(z) + g_k(z)$$

where g_k is holomorphic near z_k and $P_k(z)$ is a polynomial in $1/(z - z_k)$, in particular a rational function. Similarly we can write

$$f(1/z) = P_\infty(z) + g_\infty(z)$$

where P_∞ is a polynomial in $1/z$ and g is holomorphic at 0. Then define

$$H(z) = f(z) - P_\infty(1/z) - \sum_{k=1}^{n} P_k(z)$$

where $P_\infty(1/z)$ is a polynomial and $P_k(z)$ are polynomials in $1/(z - z_k)$. Now H is entire and $H(1/z)$ is also bounded in a punctured neighborhood of the origin. In particular H is entire and bounded and hence constant by Liouville.

Note one consequence: rational functions are determined by their principal parts at their poles. Rational functions with poles at $\{z_1, \dots, z_n\}$ (possibly repeated, possibly ∞) is a vector space of dimension $n+1$.

1.21. **Riemann sphere.** . Very brief intro will think more about it later. The extended plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ can be thought of as a sphere. Show the stereographic projection.

1.22. **Argument principle.** Eventually we want to talk about

$$\log f(z) = \log |f(z)| + i\arg(f(z))$$

the argument function is multi-valued so we will need to treat this carefully.

However the logarithmic derivative is a well defined meromorphic function

$$\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)}$$

(This must be the derivative of any branch of $\log f(z)$ based on the identity $e^{\log f(z)} = z$) which has poles at the zeros of f.

The contour integral on a curve γ

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz$$

can be interpreted as the change of the argument along the curve γ.

If f has a zero at z_0 of order m then

$$f(z) = (z - z_0)^m g(z)$$

and

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}$$

where g'/g is holomorphic near z_0 since g is nonzero there.

Similarly at a pole $f(z) = (z - z_0)^{-m} h(z)$

$$\frac{f'(z)}{f(z)} = \frac{-m}{z - z_0} + \frac{h'(z)}{h(z)}.$$

Thus the residue of f'/f at a zero or pole of f is

$$\mathrm{Res}(f, z_0) = m$$

where $m \in \mathbb{Z}$ is the order of the zero / negative order of the pole.

(Argument Principle): If f is meromorphic in Ω and D is a disk contained in Ω with $C = \partial D$ oriented counter-clockwise then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz = N_z - N_p$$

where N_z is the number of zeros of f in D and N_p is the number of poles in D (both counted with multiplicity).

Corollary: same result holds for toy contours.

This is an example of a *topological invariant*. Any curve which is homotopic to C in the complement of the set $Z \cup P$ of zeros and poles of f will also have the same argument integral.

Let's define a related notion now of *winding number* the number of times a closed curve γ winds around a point z

$$W_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} d\zeta$$

For example if $\gamma(t) = e^{2\pi i k t}$ for $t \in [0, 1]$ can explicitly compute $W_\gamma(0) = k$.

Note, by residue theorem, if γ is a positively oriented toy contour (in particular simple closed) then $W_\gamma(z) = 1$ for all z inside of γ.

Lemma: If γ is a piecewise smooth closed curve in $\mathbb{C}$ and $z \notin \gamma$ then $W_\gamma(z) \in \mathbb{Z}$. Further $W_\gamma(z)$ is constant on connected components of the complement of γ, and zero on the unbounded component of the complement of γ.

proof: Parametrize γ over $[0, 1]$ and consider

$$G(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s) - z} \, ds$$

then G is continuous and differentiable (except at finitely many points). Consider

$$H(t) = (\gamma(t) - z) e^{-G(t)}$$

which has then

$$H'(t) = \gamma'(t) e^{-G(t)} - (\gamma(t) - z) G'(t) e^{-G(t)} = \gamma'(t) e^{-G(t)} - (\gamma(t) - z) \gamma'(s) (\gamma(t) - z)^{-1} e^{-G(t)} = 0.$$

So since γ is closed

$$(\gamma(0) - z) e^{-G(0)} = H(0) = H(1) = (\gamma(1) - z) e^{-G(1)}$$

or

$$1 = e^{-G(0)} = e^{-G(1)}$$

so $G(1)$ is a multiple of $2\pi i$. (Note this proof is using that $1/(\zeta - z)$ is the derivative of a logarithm so integrating on a closed curve the result must be zero modulo $2\pi i$).

Note that W_γ is continuous and integer valued on each connected component of the complement of γ and $W_\gamma(z) \to 0$ as $z \to \infty$.

Can interpret argument principle as the *number of times* $f(\gamma)$ *winds around* 0 note that

$$\int_{f(\gamma)} \frac{1}{w} dw = \int_0^1 \frac{1}{f(\gamma(t))} f'(\gamma(t))\gamma'(t)dt = \int_\gamma \frac{f'(z)}{f(z)} dz$$

(Rouche's theorem) Suppose that f and g are holomorphic on an open set containing a toy contour γ and its inside U and

$$|f(z)| > |g(z)| \quad \text{on} \quad \gamma$$

then f and $f + g$ have the same number of zeros on U.

Proof: (Draw a picture of $f(\gamma)$ winding around 0 and the affect of perturbing by g). The proof is by continuation we claim that

$$f_t(z) = f(z) + tg(z)$$

has the same number of zeros inside of U for all $t \in [0, 1]$. Note that

$$N(t) = \frac{1}{2\pi i} \int_\gamma \frac{f_t'(z)}{f_t(z)} \, dz$$

since $|g(z)| < |f(z)| - \delta$ on γ

$$|f_t(z)| \geq |f(z)| - t|g(z)| \geq \delta \quad \text{for all} \quad t \in [0, 1]$$

so $f_t(z)$ is jointly continuous in (t, z) on $[0, 1] \times \gamma$ and so we can argue that $N(t)$ is continuous in t, more specifically

$$|N(t) - N(s)| = \frac{1}{2\pi i} | \int_\gamma \frac{(t-s)g'(z)}{f_t(z)} + f_s'(z) \frac{f_s(z) - f_t(z)}{f_t(z) f_s(z)} \, dz| = \frac{1}{2\pi i} | \int_\gamma \frac{(t-s)g'(z)}{f_t(z)} + f_s'(z) \frac{(t-s)g(z)}{f_t(z) f_s(z)} \, dz|$$

However by argument principle $N(t)$ is integer valued so $N(t)$ must be constant on $[0, 1]$ meaning $N(0) = N(1)$.

Examples: Fundamental theorem of algebra via Rouche, prove $z^5 + 3z^3 + 7$ has all zeros in $|z| < 2$, $e^z - z$ in the unit disk.

(Open Mapping Theorem): If f is holomorphic and nonconstant in a region Ω then f is open (images of open sets are open).

Proof: Let $w_0 = f(z_0)$ be in the image of f we want to show that a neighborhood of w_0 is in $f(\Omega)$ as well. Let w near w_0 (to be specified) and consider $g(z) = f(z) - w$.

Write

$$g(z) = (f(z) - w_0) + (w_0 - w) = F(z) + G(z)$$

F has a zero of some order $m \geq 1$ at z_0 (finite order since f nonconstant). Then $F(z) = (z - z_0)^m H(z)$ with H holomorphic and nonzero in a neighborhood $\overline{D_r(z_0)}$ so $|F(z)| \geq \delta > 0$ on $\partial D_r(z_0)$. So if $0 < |w - w_0| < \delta$ then $|G(z)| < |F(z)|$ on $\partial D_r(z_0)$ so by Rouche's theorem $F + G$ has the same number of zeros as F in $D_r(z_0)$ (which is $m \geq 1$). Thus w is in the image $f(D_r(z_0))$.

(Maximum modulus principle): If f is holomorphic in a domain Ω and non-constant then $|f|$ cannot attain its maximum in Ω.

Proof: If $|f(z_0)| = \max_\Omega |f(z)|$ for some $z_0 \in \Omega$ then by the open mapping theorem $D_r(f(z_0)) \subset f(\Omega)$ but there is a point in $D_r(f(z_0))$ with strictly larger magnitude that $f(z_0)$, $f(z_0) + \frac{r}{2} f(z_0)/|f(z_0)|$.

Corollary: Holomorphic functions on compact sets attain their maximum on the boundary of the set.

(Cautions about unbounded sets)

1.23. **Complex Logarithm.** As we have discussed in passing if we want to define the logarithm of a complex number $z = re^{i\theta}$ it is natural to define

$$\log z = \log r + i\theta$$

the problem is that θ is unique only up to integer multiples of 2π.

However locally near any $z_0 \neq 0$, fixing a particular value of θ_0, we can extend $\log z$ to be defined as a holomorphic function in a neighborhood. Different choices of the base value θ_0 will result in different values of the logarithm differing by $2\pi i$, these are called **branches** of the argument or of the logarithm.

Theorem: If Ω is simply connected with $1 \in \Omega$ and $0 \notin \Omega$ there there is a branch of the logarithm $F(z) = \log_\Omega(z)$ so that (1) F holomorphic on Ω, (2) $e^{F(z)} = z$ on Ω and $F(r) = \log(r)$ whenever r is real and near 1.

Proof: Construct $\log_\Omega z$ as the primitive of $\frac{1}{z}$ based at 1. Need to check that $ze^{-F(z)} = 1$, do this by computing derivative.

Comment about spiral like domain for why logarithm may not be real on the entire positive real axis.

Principal branch of the Logarithm defined on $\mathbb{C} \setminus (\infty, 0]$ argument takes values in $(-\pi, \pi)$

$$\log(z) = \log r + i\theta$$

Proof that this is the same as the logarithm defined before: integrate from 1 to r along the real axis then integrate along an arc to $re^{i\theta}$.

Note logarithm can fail to satisfy $\log(z_1 z_1) = \log(z_1) + \log(z_2)$ check with $e^{2\pi i/3}$.

Taylor series expansion for principal branch

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

find by integrating taylor series of $\frac{1}{1+z}$ term by term. Both have the same derivative and agree at $z = 1$.

Fractional powers: Ω is a simply connected domain containing 1 and not 0 and $\log_\Omega z$ is the branch of the logarithm defined on Ω with $\log(1) = 0$ then can define

$$z^\alpha = e^{\alpha \log z}$$

check that $1^\alpha = 0$ and $(z^{1/n})^n = 1$

Theorem: If f is a nowhere vanishing holomorphic function on a simply connected domain Ω then there is a holomorphic function g on Ω so that

$$f(z) = e^{g(z)}$$

This is a branch of $\log f(z)$.

proof: Fix z_0 and any value c_0 with $e^{c_0} = f(z_0)$. We define the logarithm by integrating the logarithmic derivative $\frac{f'(z)}{f(z)}$

$$g(z) = \int_{\gamma_z} \frac{f'(z)}{f(z)}\, dz + c_0$$

Compute the derivative of $f(z)e^{-g(z)}$.

Contour integral of $\int_0^\infty \frac{\log(x)}{x^2+a^2}$.

Riemann surface M: connected complex manifold of dimension 1: connected Hausdorff topological space, every point on M has a neighborhood which is homeomorphic to the open unit disc in $\mathbb{C}$, and there is an atlas of local charts (i.e. an open cover of M by U_α which each come with a chart $\varphi_\alpha : U_\alpha \to V_\alpha$ which is a homeomorphism from an open subset of $\mathbb{C}$) and the transition maps $\tau_{\alpha,\beta} = \varphi_\alpha^{-1} \circ \varphi_\beta : V_\alpha \to V_\beta$ are all holomorphic.

Specifics of this definition will not really be used, for us this is mainly an issue of intuition about analytic continuation.

Examples:

- Subsets of $\mathbb{C}$ - The Riemann sphere. - Riemann surface of the logarithm - Riemann surface of $z^{1/2}$ - Riemann surface of $z^{1/3}$

1.24. **Full Laurent series.** Topic was not in the book but let's mention it: Function f has isolated singularity at 0, think of essential singularity case (only one we have not studied yet). Do the Laurent series expansion in an annulus

$$f(z) = \int_{D_R(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{D_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta = F(z) - G_r(z).$$

Note that $F(z)$ and $f(z)$ are independent of r for $|z - z_0| > r$ so $G_r(z)$ must be as well. Expand $F(z)$ as a power series z convergent in $D_R(z_0)$. Expand $G_r(z)$ in a power series in $\frac{1}{z}$ convergent in $|z - z_0| > r$. Call the coefficients $a_n(r)$ for $n \leq -1$. Note coefficients cannot depend on r because $G_r(z) = G_{r'}(z)$ on $|z - z_0| \geq \max\{r, r'\}$ so

$$0 = \sum_{-\infty}^{-1} (a_n(r) - a_n(r'))(z - z_0)^n \quad \text{on} \quad |z - z_0| > \max\{r, r'\}.$$

Multiply by $(z - z_0)$ and send $z \to \infty$ to find $a_{-1}(r) = a_{-1}(r')$, then proceed to a_{-2} etc.

Uniqueness of Laurent series expansion: Given a series expansion

$$f(z) = \sum_k a_k z^k$$

converging in a punctured neighborhood $\overline{D(0,r)} \setminus \{0\}$ simple compute

$$\frac{1}{2\pi i}\int_{\partial D(0,r)} z^{-k-1}f(z)\ dz = a_k.$$

2. Conformal mappings

A **conformal mapping** $f : U \to V$ is a bijective holomorphic function. If there is a conformal mapping between U and V we say the sets are **conformally equivalent**. Note that this is indeed an equivalence relation because:

Lemma. If $f : U \to V$ is conformal then $f'(z) \neq 0$ for all $z \in U$ and the inverse of f defined on V is holomorphic.

Proof: If $f'(z_0) = 0$ then $f(z) = a(z - z_0)^m + G(z)$ for some $m \geq 2$ and then apply Rouche's theorem on a small disk. (Exactly proof of open mapping theorem).

Conformal mappings are **angle preserving** if two curves γ and η intersect at z_0 then $f\circ\gamma$ and $f\circ\eta$ intersect at $f(z_0)$ and the angles between the tangent vectors agree. Euclidean inner product $(z, w) = \mathrm{Re}(z\bar{w})$ and the cosine of the angle between two vectors is $(z, w)/|z||w|$.

2.1. **Disk and upper half-plane.**

Conformal equivalence of disk $\mathbb{D}$ and upper half plane $\mathbb{H} = \{\mathrm{Im}(z) > 0\}$.

Explicit mappings

$$\phi(z) = \frac{i-z}{i+z} \quad \text{and} \quad \psi(w) = i\frac{1-w}{1+w}.$$

The mapping ϕ is conformal $\mathbb{H} \to \mathbb{D}$ and ψ is its inverse.

Note both maps are holomorphic on their domains. ψ has a singularity at -1 which gets mapped to ∞ (which is on the boundary of the upper half space viewed as a subset of the Riemann sphere. (Point out that disk and upper half space are both half spheres under the stereographic projection).

Note that any point in $\mathbb{H}$ is closer to i than to $-i$ so $|\phi(z)| < 1$ on $\mathbb{H}$.

Also ψ maps $\mathbb{D}$ to $\mathbb{H}$ again by direct computation

$$\mathrm{Im}(\psi) = \mathrm{Re}(\frac{1-w}{1+w}) = \mathrm{Re}(\frac{(1+\bar{w})(1-w)}{(1+\bar{w})(1+w)}) = \frac{1-|w|^2}{|1+w|^2}$$

Finally $\phi(\psi(w)) = w$ so ϕ is onto $\mathbb{D}$ and similarly ψ is onto $\mathbb{H}$.

Also point out boundary behavior by evaluating on boundaries.

2.2. **More examples.**

Translations, dilations/rotations, mapping sector $\{|\arg(z)| < \pi/2n\}$ to right half plane via z^n, inverse mapping via $z^{1/n}$, point out the boundary behavior at a corner. Logarithm maps upper half plane to strip domain, maps upper half disc to half-strip domain.

Mapping upper half disc to quarter-plane via $f(z) = \frac{1+z}{1-z}$ (multiply num/denom by $1 - \bar{z}$ and compute real/imaginary parts) with inverse $g(w) = \frac{w-1}{w+1}$.

Mapping from slit disk to disk, pictures: square root, rotate, map to quarter plane, square, map to disc.

2.3. **Automorphisms of the disk and Schwarz Lemma.** Schwarz Lemma: Let $f : \mathbb{D} \to \mathbb{D}$ holomorphic with $f(0) = 0$ then (i) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$, (ii) If for some $z_0 \neq 0$ $|f(z_0)| = |z_0|$ then f is a rotation, (iii) $|f'(0)| \leq 1$ and if equality holds then f is a rotation.

proof: Expand in power series $f(z) = a_1 z + a_2 z^2 + \cdots$. So $f(z)/z$ is holomorphic in $\mathbb{D}$. Then $|f(z)/z| \leq \frac{1}{r}$ on $|z| = r$. By maximum modulus principle $|f(z)/z| \leq \frac{1}{r}$ for $|z| \leq r$. Sending $r \to 1$ gives the first result. For the second result if $|f(z)/z|$ attains its maximum at an interior point then it is constant meaning f is a rotation. Finally for the third part notice that $\lim_{z\to 0} f(z)/z = f'(0)$ so we get $|f'(0)| \leq 1$. If $|f'(0)| = 1$ then $g(z) = f(z)/z$ attains its maximum modulus at 0 which again would imply f is a rotation.

A conformal map of Ω to itself is called an *automorphism*. The set $\mathrm{Aut}(\Omega)$ is a group under the operation of map composition with the identity map being the group identity.

For the unit discs rotations are automorphisms as are the maps

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z} \quad \text{For} \quad \alpha \in \mathbb{D}.$$

We saw on HW 1 that these maps are bijections of $\mathbb{D}$ to itself with $\psi_\alpha \circ \psi_\alpha = id$. Can think of ψ_α as defined by the property that it is an automorphism of $\mathbb{D}$ which exchanges α and 0.

In fact rotations and these Blaschke factor exchanges are the only disk automorphisms: If f is a disk automorphism then

$$f(z) = e^{i\theta}\psi_\alpha(z)$$

for some $\theta \in \mathbb{R}$ and some $\alpha \in \mathbb{D}$.

Proof: Since f is an automorphism there is a unique α such that $f(\alpha) = 0$. Precompose with ψ_α so that f fixes the origin

$$g(z) = (f \circ \psi_\alpha)(z)$$

and we can apply Schwarz Lemma

$$|g(z)| \leq |z|$$

and also to g^{-1}

$$|g^{-1}(w)| \leq |w|.$$

However this implies with $w = g(z)$

$$|z| \leq |g(z)|$$

so $|g(z)| = |z|$ for all $z \in \mathbb{D}$ and so by Schwarz Lemma g is a rotation.

$$f(\psi_\alpha(z)) = e^{i\theta} z$$

and then plugging in $z = \psi_\alpha(w)$ we find

$$f(w) = e^{i\theta}\psi_\alpha(w)$$

using ψ_α is its own inverse.

Note: Corollary is "Any disk automorphism which fixes the origin is a rotation".

2.4. **Riemann Mapping Theorem.** Observations: conformal maps take simply connected regions to simply connected regions, the disk cannot be conformally equivalent to $\mathbb{C}$ because of Liouville's theorem.

Theorem: If Ω is proper and simply connected and $z_0 \in \Omega$ then there is a unique conformal map $\phi : \Omega \to \mathbb{D}$ so that $\phi(z_0) = 0$ and $\phi'(z_0) > 0$.

We will consider the class of all injective holomorphic maps from Ω to $\mathbb{D}$ and then maximizing $|f'(z_0)|$ will turn out to suffice to make the map surjective. (Think in rough analogy to Schwarz Lemma). The key issue here is how to show that there EXISTS an injective holomorphic map which achieves the supremal value for $|f'(z_0)$ in this class.

2.5. **Montel and Hurwitz Theorem.** Let Ω be a domain in $\mathbb{C}$. A family $\mathcal{F}$ of holomorphic functions is called **normal** if every sequence in $\mathcal{F}$ has a subsequence converging uniformly on compact subsets of Ω. The limit does not need to be an element of $\mathcal{F}$. In real analysis terminology we would say that this family is **precompact** in the topology of local uniform convergence on Ω.

We say that a family $\mathcal{F}$ is uniformly bounded on compact sets of Ω if for all $K \subset\subset \Omega$ there is B so that

$$|f| \leq B \quad \text{for all } f \in \mathcal{F}.$$

Theorem (Montel): If $\mathcal{F}$ is a family of holomorphic functions which is uniformly bounded on compact subsets of Ω then: (i) $\mathcal{F}$ is equicontinuous on compact subsets of Ω, (ii) $\mathcal{F}$ is a normal family.

Proved on homework 2.

Theorem (Hurwitz): If Ω connected and f_n injective on Ω converge locally uniformly on Ω to f then f is either injective or constant.

Proof: Assume f is not constant, and $f(z_1) = f(z_2)$. Look at $g_n(z) = f_n(z) - f_n(z_1)$ which have exactly one zero at z_1. Then use argument principle on a small circle around z_2 to show that g cannot have a zero at z_2.

2.6. **RMT proof.** . Step 1: Simply connected proper domain is conformal to an open subset of $\mathbb{D}$. Define a branch of the logarithm $e^{f(z)} = z - \alpha$ where $\alpha \notin \Omega$. The map is injective by using the formula $z - \alpha = e^{f(z)} = e^{f(w)} = w - \alpha$. Fixing a $w \in \Omega$ the value $f(w) + 2\pi i$ cannot be taken on Ω, actually $|f(z) - (f(w) + 2\pi i)|$ is bounded from below on Ω. Again follows by applying the formula $z - \alpha = e^{f(z)} = e^{f(w)+2\pi i} = w - \alpha$, also to a sequence with $f(z_n) \to f(w) + 2\pi i$. Then take

$$\varphi(z) = \frac{1}{f(z) - (f(w) + 2\pi i)}.$$

Step 2. Assume $\Omega \subset \mathbb{D}$ and contains 0 and define

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} : f \text{ holomorphic, injective, and } f(0) = 0\}$$

Take a sequence maximizing $|f'(0)|$ over $\mathcal{F}$ (supremum is at least 1 due to identity being in $\mathcal{F}$). Apply Montel and Hurwitz to show there is a uniformly convergent subsequence and the limit is an element of $\mathcal{F}$ achieving the supremum.

Step 3. Suppose the constructed f is not surjective, $f(\Omega) = \tilde{U} \subset \mathbb{D}$ but there is $\alpha \in \mathbb{D} \setminus \tilde{U}$ ($\tilde{U}$ is simply connected). Construct another map with larger derivative at 0. First map by ψ_α so $U = \psi_\alpha(f(\Omega))$ misses 0. Then define a branch $g(w) = e^{\frac{1}{2}\log(w)}$ of the square root on U. Then create a new map

$$F = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f$$

Check that $F \in \mathcal{F}$.

Then

$$f = \psi_\alpha^{-1} \circ h \circ \psi_{g(\alpha)}^{-1} \circ F = \Phi \circ F$$

where Φ is a self-map of the disc with $\Phi(0) = 0$ which is not injective because h is not injective. Thus $|\Phi'(0)| < 1$ be Schwarz Lemma and

$$f'(0) = \Phi'(F(0))F'(0) = \Phi'(0)F'(0)$$

so $|f'(0)| < |F'(0)|$ which is a contradiction.

2.7. **Fractional linear transformations.** Give $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ we call

$$F(z) = \frac{az + b}{cz + d}$$

to be a **fractional linear transformation** (mapping would be trivial if the determinant mentioned were zero). We have already seen examples of this type, sub-families which make up the conformal self-maps of $\mathbb{D}$ and $\mathbb{H}$.

Each transformation F is associated with a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with nonzero determinant. Note that multiplication of each matrix entry by $\lambda \in \mathbb{C} \setminus \{0\}$ gives rise to the same mapping so we could add the normalization $ad - bc = 1$.

Vice versa if two matrices give rise to the same transformation they must be multiples of each other.

Note

$$F'(z) = \frac{ad - bc}{(cz + d)^2}$$

This function has a pole at $-d/c$ but is nonzero on $\mathbb{C} \setminus \{-d/c\}$. Thus F is locally conformal away from the pole.

F has an inverse

$$w = \frac{az + b}{cz + d}$$

then

$$czw + dw = az + b$$

and

$$z = \frac{dw - b}{-cw + a}$$

i.e. the matrix

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

which is actually the matrix inverse (when the determinant is normalized to be 1).

Thus F is a conformal map from $\mathbb{C} \setminus \{-d/c\}$ to $\mathbb{C} \setminus \{a/c\}$.

This becomes much more intuitive if we allow ourselves to view F as a conformal self-map of the Riemann sphere / extended complex plane, i.e. $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

Then we view

$$F(\infty) = a/c \;\text{ if }\; c \neq 0$$

$$F(\infty) = \infty \;\text{ if }\; c = 0$$

and

$$F(-d/c) = \infty \;\text{ if }\; c \neq 0.$$

Then we view F as a meromorphic bijection of $\mathbb{C}^*$ with itself, which we may also view as a conformal mapping of the sphere.

Note that fractional linear transformations are made up of compositions of translations, inversions and dilations/rotations.

Fractional linear transformations have a lot of nice properties:

Theorem: Fractional linear transformations map straight lines and circles in $\mathbb{C}^*$ onto straight lines and circles.

A straight line on $\mathbb{C}^*$ is simply a straight line in $\mathbb{C}$ union with the point at infinity. It is helpful to think of straight lines as *circles through* ∞. Then the statement is simply that fractional linear transformations map circles to circles.

Proof: FLT are compositions of translations, inversions and rotations/dilations so we just need to check the property for each of these. The only one which is not obvious is inversions.

Consider the map $z \mapsto 1/z = u + iv$ with

$$u(x) = \frac{x}{x^2+y^2} \;\text{ and }\; v(x) = \frac{-y}{x^2+y^2}.$$

The equation of a straight line or circle in the (u, v)-plane has the form

$$A(u^2 + v^2) + Bu + Cv = D$$

for some real numbers A, B, C, D not all zero.

Plugging in the values of (u, v) in terms of (x, y) we find

$$A + Bx - Cy = D(x^2 + y^2)$$

which is also the equation of a line or circle. Thus the inverse image of a circle is a circle under inversion (which is its own inverse). Lines are mapped to circles through 0, circles through zero are mapped to lines and remaining circles are mapped to circles.

Fixed points of FLT are just z_0 such that $F(z_0) = z_0$. If ∞ is a fixed point of a FLT F then F is linear: if $c \neq 0$ then $F(\infty) = a/c$ which is not infinity unless $c = 0$.

FLT can be determined by their behavior on 3 points,

Theorem: Given any (z_1, z_2, z_3) distinct on the extended plane and (w_1, w_2, w_3) distinct there is a unique FLT with

$$F(z_i) = w_i$$

Proof: Suppose F and G are both FLT mapping $z_j \to w_j$ then $F \circ G^{-1}$ fixes (z_1, z_2, z_3) so we can just show

Lemma: If F is a FLT with three fixed points then F is identity.

Proof: First suppose $z_3 = \infty$ then $F(z) = az + b$. If $z_1 \in \mathbb{C}$ is a fixed point then $az_1 + b = z_1$ so $(1 - a)z_1 = b$. If $a \neq 1$ then $z_1 = b/(1 - a)$ is the only possible fixed point in $\mathbb{C}$. If $a = 1$ then $z + b$ has a fixed point if and only if $z = 0$ in which case F is identity.

If ∞ is not a fixed point so $c \neq 0$ then

$$\frac{az + b}{cz + d} = z$$

and

$$cz^2 + (d - a)z - b = 0$$

the equation has at most two roots so F has at most two fixed points (roots from quadratic formula).

Finding the map explicitly: first figure out how to send $(z_1, z_2, z_3) \mapsto (0, \infty, 1)$

$$z \mapsto \frac{z - z_1}{z - z_2}$$

has the correct behavior for z_1, z_2 then

$$\varphi(z) = \frac{z_3 - z_2}{z_3 - z_1} \frac{z - z_1}{z - z_2}$$

is the unique FLT sending (z_1, z_2, z_3) to $(0, \infty, 1)$. The quantity

$$(z_1, z_2; z_3, z_4) = \frac{z_3 - z_2}{z_3 - z_1} \frac{z_4 - z_1}{z_4 - z_2}$$

is known as the **cross-ratio**.

If F is the FLT mapping $(z_1, z_2, z_3) \mapsto (w_1, w_2, w_3)$ then

$$F(z) = \varphi^{-1}_{(w_1,w_2,w_3)}(\varphi_{(z_1,z_2,z_3)}(z))$$

so

$$\varphi_{(w_1,w_2,w_3)}(F(z)) = \varphi_{(z_1,z_2,z_3)}(z)$$

i.e.

$$\frac{F(z_3) - F(z_2)}{F(z_3) - F(z_1)} \frac{F(z_4) - F(z_1)}{F(z_4) - F(z_2)} = \frac{z_3 - z_2}{z_3 - z_1} \frac{z_4 - z_1}{z_4 - z_2}$$

the cross-ratio is preserved by fractional linear transformations. (This should not be memorized, simply remember how to map to $(0, 1, \infty)$)

3. Entire functions

(Following Stein-Shakarchi Chapter 5) Questions: Zero sets, growth at infinity, factorization based on the zeros.

3.1. **Jensen's formula.**

. Number of zeros of polynomial is exactly related to the polynomial growth rate at infinity.

Theorem: Let Ω be a domain containing $\overline{D}_R$ and f holomorphic on Ω, $f(0) \neq 0$ and f vanishes nowhere on C_R. If $z_1, \ldots, z_N$ are the zeros of f counted with multiplicity then

$$|\log f(0)| = \sum \log \frac{|z_k|}{R} + \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta$$

Proof: If f_1 and f_2 satisfy the theorem then $f_1 f_2$ also does because the formula $\log(ab) = \log a + \log b$ holds for positive real inputs and the zero set of $f_1 f_2$ is the union of the two zero sets.

$f(z) = g(z)\Pi(z - z_j)$ where g is holomorphic and non-vanishing so we can prove the result for nonvanishing g and for monomials.

First for g nonvanishing: g has a holomorphic logarithm on a nbhd of D_R so $g(z) = e^{h(z)}$ and $|g(z)| = e^{\mathrm{Re}(h(z))}$ so $\log |g(z)| = \mathrm{Re}(h(z))$. Since $\Re(h(z))$ is harmonic it satisfies the mean value property

$$\mathrm{Re}(h(0)) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}(h(z)) d\theta.$$

For a factor $z - w$ for some fixed $w \in D_R$ need to show

$$\log |w| = \log \frac{|w|}{R} + \frac{1}{2\pi} \int_0^{2\pi} \log |Re^{i\theta} - w| d\theta.$$

or

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \log |e^{i\theta} - \frac{w}{R}| d\theta$$

so it suffices to show

$$\int_0^{2\pi} \log |e^{i\theta} - a| d\theta = 0 \quad \text{for} \quad |a| < 1$$

or

$$\int_0^{2\pi} \log |1 - ae^{i\theta}| d\theta = 0$$

(changing variables $\theta \to -\theta$). The function $F(z) = 1 - az$ does not vanish on $\mathbb{D}$ so it has a holomorphic logarithm $F(z) = e^{H(z)}$ there and $\log |F| = \mathrm{Re}(H)$ and $F(0) = 1$ so $\log |F(0)| = 0$ so MVT again

$$0 = \mathrm{Re}(H(0)) = \frac{1}{2\pi} \int_0^{2\pi} \log \mathrm{Re}(H(e^{i\theta})) d\theta.$$

That completes the proof of Jensen's formula.

This is going to give us a way to connect values of f with the number of zeros of f in a given disc:

Lemma:

$$\int_0^R \mathfrak{n}(r)\frac{dr}{r} = \sum \log\left|\frac{R}{z_k}\right|.$$

where $\mathfrak{n}(r)$ is the number of zeros of f inside of D_R.

proof: note

$$\sum \log\left|\frac{R}{z_k}\right| = \sum \int_{|z_k|}^R \frac{dr}{r} = \sum \int_0^R \mathbf{1}_{|z_k|>r}\frac{dr}{r} = \int_0^R \sum_k \mathbf{1}_{r>|z_k|}\frac{dr}{r}$$

3.2. **Functions of finite order.** Let f entire if

$$|f(z)| \leq Ae^{B|z|^\rho} \quad \text{for all} \quad z \in \mathbb{C}$$

we say that f has order of growth $\leq \rho$ and the order of growth of f is

$$\rho_f = \inf \rho$$

over all ρ so that f has order at most ρ.

Theorem: If f is entire with order of growth $\leq \rho$ then

$$\mathfrak{n}(r) \leq Cr^\rho$$

for large r and if $z_k \neq 0$ are the zeros of f then for all $s > \rho$

$$\sum_k \frac{1}{|z_k|^s} < +\infty.$$

Proof: Without loss we can assume $f(0) \neq 0$ (otherwise divide by the z^ℓ order of the zero) only affects $\mathfrak{n}(r)$ by a constant and doesn't change the order).

Then apply Jensen's formula in the form

$$\int_0^R \mathfrak{n}(x)\frac{dx}{x} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|$$

and take $R = 2r$

$$\int_r^{2r} \mathfrak{n}(x)\frac{dx}{x} \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|$$

since $\mathfrak{n}$ is increasing

$$\int_r^{2r} \mathfrak{n}(x)\frac{dx}{x} \geq \mathfrak{n}(r)\log 2$$

while

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \log|Ae^{BR^\rho}|d\theta \leq Cr^\rho.$$

For the second part of the theorem

$$\sum |z_k|^{-s} = \sum_j \sum_{2^j \leq |z_k| < 2^{j+1}} |z_k|^{-s} \leq \sum_j 2^{-js}\mathfrak{n}(2^{j+1}) \leq C\sum_j 2^{-js}2^{(j+1)\rho} < +\infty$$

Examples: $f(z) = \sin \pi z$ which has $|f(z)| \leq e^{\pi|z|}$ also f has simple zeros at each $n \in \mathbb{Z}$ and

$$\sum_{n \neq 0} \frac{1}{|n|^s} < +\infty$$

if and only if $s > 1$.

$f(z) = \cos(z^{1/2})$ defined by the power series

$$\sum_{n=0}^{\infty} (-1)^n z^n/(2n)!.$$

Then $f(z)$ is entire and $|f(z)| \leq e^{|z|^{1/2}}$ has order of growth $1/2$ zeros at $((n+1/2)\pi)^2$ summable exactly for $s > 1/2$.

3.3. **Infinite products.** Next goal is to try to find a factorization formula by the zeros of an entire function in the case when f is not just a polynomial.

Given a sequence $a_n \in \mathbb{C}$ we say that the product $\prod_{n=1}^{\infty}(1+a_n)$ converges if the limit of the partial products exists:

$$\lim_{N \to \infty} \prod_{n=1}^{N} (1 + a_n).$$

Lemma: If $\sum |a_n| < \infty$ then the product $\prod(1 + a_n)$ converges and the limit is zero if and only if one of the factors is zero.

Since $\sum |a_n|$ converges $|a_n| < 1/2$ for sufficiently large n, in fact we can assume it is true for all n by factoring out a finite product. Then we can use the standard power series definition for $\log(1 + z)$ which converges in $D(0, 1)$ so

$$\prod_{1}^{N} (1 + a_n) = \prod_{1}^{N} e^{\log(1+a_n)} = e_N^B$$

where

$$B_N = \sum_{n=1}^{N} b_n \quad \text{with} \quad b_n = \log(1 + a_n).$$

By fundamental theorem of calculus

$$|\log(1 + z)| \leq \max_{|w| \leq 1/2} \frac{1}{|1 + w|} |z| \leq 2|z|$$

so $|b_n| \leq 2|a_n|$ and B_N converges to some B. The limit is nonzero because it is e^B, so the limit could only be zero if one of the finitely many terms we cut off before was zero.

Products of holomorphic functions: If F_n are holomorphic on a domain Ω and $|F_n(z) - 1| \leq c_n$ and $\sum c_n < +\infty$ then the product $\prod F_n(z)$ converges uniformly on Ω to a holomorphic function F and if F_n does not vanish for any n then

$$\frac{F'(z)}{F(z)} = \sum \frac{F_n'(z)}{F_n(z)}.$$

Proof: Write $F_n(z) = 1 + a_n(z)$ with $|a_n| \le c_n$

$$|e^{B_N} - e^B| \le \max\{e^{B_N}, e^B\}|B_N(z) - B(z)| \le \max\{e^{B_N}, e^B\} \sum_N^\infty c_n$$

so we actually get a uniform estimate of the convergence from the tails of the dominating series $\sum c_n$.

Call

$$G_N(z) = \prod_1^N F_n(z)$$

the $G_N \to F$ uniformly on Ω so on any compact $K \subset \Omega$ the G_N' converge uniformly to F' since G_N are uniformly bounded from below on K

$$\frac{G_N'}{G_N} \to \frac{F'}{F} \text{ unif on } K.$$

Compute G_N'/G_N.

3.4. **Weierstrass product theorem.** Theorem: Given any sequence a_n of complex numbers with $a_n \to \infty$ there is an entire function f vanishing at $z = a_n$ and nowhere else (counted with multiplicity), furthermore any other such entire function can be written $f(z)e^{g(z)}$ with g entire.

If f_1 and f_2 both satisfy then f_1/f_2 has removable singularities at the a_n and does not take the value 0. Therefore it has a global logarithm.

We cannot simply take the product

$$z^m \prod_n (1 - \frac{z}{a_n})$$

where m is the order of the zero of f at 0 because the infinite product will not converge in general.

We define the **canonical factors**

$$E_0(z) = (1 - z) \text{ and } E_k(z) = (1 - z)e^{z + z^2/2 + \cdots + z^k/k}.$$

Lemma: If $|z| \le 1/2$ then $|1 - E_k(z)| \le c|z|^{k+1}$ with constant c independent of k.

Note that the power series in the exponential is a partial sum for the power series of $-\log(1 - z)$ centered at $z = 0$.

$$E_k(z) = e^{\log(1-z) + z + z^2/2 + \cdots + z^k/k} = e^{\sum_{k+1}^\infty z^\ell/\ell}$$

which has

$$|\sum_{k+1}^\infty z^\ell/\ell| \le |z|^{k+1} \sum_{k+1}^\infty |z|^{\ell-k-1}/\ell \le |z|^{k+1} \sum 2^{-j} = 2|z|^{k+1}.$$

In particular it has magnitude at most 1 and so

$$|1 - e^{\sum_{k+1}^\infty z^\ell/\ell}| \le 2e|z|^{k+1}.$$

Given zero of order m at the origin the Weierstrass product factorization is

$$f(z) = z^m \prod E_n(z/a_n).$$

We need to just check that the product converges in $D(0,R)$ for each $R > 0$ and has the correct zeros.

Let $z \in D(0,R)$ and split the zeros a_n by $a_n \in D(0,2R)$ or not. The first collection is finite, and the finite product

$$z^m \prod_{|a_n|<2R} E_n(z/a_n)$$

has the desired properties in $D(0,R)$. The remainder has

$$|1 - E_n(z/a_n)| \leq 2|z/a_n|^{n+1} \leq 2^{-n}$$

and so the infinite product

$$\prod_{|a_n|\geq 2R} E_n(z/a_n)$$

converges uniformly in $D(0,R)$ to a holomorphic function which does not vanish on that disk (since none of the factors take the value zero there).

3.5. **Hadamard product theorem.** The issue with the Weierstrass product theorem

$$f(z) = e^{g(z)} z^m \prod E_n(z/a_n)$$

is that the factors E_n have order of growth n which also keeps growing as one goes out further in the product. For functions of finite order we can refine the result and bound the order of the factors E_n which are necessary in the product factorization.

Theorem: Suppose f has order of growth ρ_0 and $k = \text{floor}(\rho_0)$. Then

$$f(z) = e^{P(z)} z^m \prod E_k(z/a_n)$$

where P is a polynomial of degree at most k.

Recall that

$$\sum_{a_n \neq 0} |a_n|^{-s} < +\infty$$

for any $s > \rho_0$. Thus we can repeat the proof of Weierstrass product theorem to show that for $|a_n| \geq 2R > R > |z|$ we have $|z/a_n| \leq 1/2$ so

$$|1 - E_k(z/a_n)| \leq 2|z/a_n|^{k+1} \leq 2|z|^{k+1}|a_n|^{-k-1}$$

and so

$$\sum_{|a_n|\geq 2R} |1 - E_k(z/a_n)| \leq 2|z|^{k+1} \sum_{|a_n|\geq 2R} |a_n|^{-k-1} < +\infty$$

and so the infinite product

$$\prod_{|a_n|\geq 2R} E_k(z/a_k)$$

converges uniformly to a holomorphic function on $D(0,R)$ with no zeros on $D(0,R)$. This shows that

$$h(z) = z^m \prod E_k(z/a_n)$$

is an entire function with zeros exactly at a_n and a zero of order m at the origin. The remaining thing is to understand

$$\frac{f(z)}{h(z)} = e^{g(z)}.$$

The aim is to show that for any $k+1 > s > \rho_0$

$$|h(z)| \geq Ae^{-B|z|^s} \quad \text{on} \quad |z| = r_m$$

for some sequence of radii $r_m \to \infty$ (we need this freedom because the result would be false on radii where h has a zero, we need to choose the radius to avoid the zeros of f by a sufficiently large margin). In that case we would combine with the growth order of f to find

$$e^{\mathrm{Re}(g(z))} = \left|\frac{f(z)}{g(z)}\right| \leq A'e^{B'|z|^s} \quad \text{on} \quad |z| = r_m$$

so

$$\mathrm{Re}(g(z)) \leq B'|z|^s \quad \text{on} \quad |z| = r_m$$

with sequence $r_m \to \infty$. This implies that g is a polynomial of order at most floor$(s) = k$ by a Liouville type argument.

To achieve the needed lower bounds of the product $h(z)$ we will need some Lemmas.

Lemma: The canonical factors satisfy

$$|E_k(z)| \geq e^{-c|z|^{k+1}} \quad \text{if} \quad |z| \leq 1/2$$

and

$$|E_k(z)| \geq |1-z|e^{-c'|z|^k} \quad \text{if} \quad |z| \geq 1/2.$$

Proof: Recall

$$E_k(z) = e^{-\sum_{k+1}^{\infty} z^n/n} = e^w$$

for $|z| \leq 1/2$ so

$$|E_k(z)| = |e^w| \geq e^{-|w|}$$

and $|w| \leq |z|^{k+1}(\sum 2^{-n}/(n+k+1)) \leq 2|z|^{k+1}$.

For the second part

$$|E_k(z)| = |1-z||e^{z+z^2/2+\cdots+z^n/n}| \geq |1-z|e^{-|z+z^2/2+\cdots+z^k/k|} \geq |1-z|e^{-|z|^k}.$$

Lemma: For any s with $\rho_0 < s < k+1$ we have

$$|\prod E_k(z/a_n)| \geq e^{-c|z|^s}$$

except possibly when z belongs to the union of discs centered at a_n of radius $|a_n|^{-k-1}$.

Proof: Divide product

$$\prod E_k(z/a_n) = \prod_{|a_n|\le 2|z|} E_k(z/a_n) \prod_{|a_n|>2|z|} E_k(z/a_n)$$

and argue separately.

For the "outside" zeros

$$|\prod_{|a_n|>2|z|} E_k(z/a_n)| \ge \prod_{|a_n|>2|z|} e^{-c|z/a_n|^{k+1}} = e^{-c|z|^{k+1}\sum_{|a_n|>2|z|}|a_n|^{-k-1}}$$

But

$$\sum_{|a_n|>2|z|} |a_n|^{-k-1} = \sum_{|a_n|>2|z|} |a_n|^{-s}|a_n|^{s-k-1} \le \sum_{|a_n|>2|z|} |a_n|^{-s}|2z|^{s-k-1} = C|z|^{s-k-1}.$$

using the summability of $|a_n|^{-s}$ (via Jensen's Lemma and corollaries earlier).

For the "inside" zeros

$$|\prod_{|a_n|\le 2|z|} E_k(z/a_n)| \ge \prod_{|a_n|\le 2|z|} |1 - z/a_n| \prod_{|a_n|\le 2|z|} e^{-c|z/a_n|^k}$$

and

$$\prod_{|a_n|\le 2|z|} e^{-c|z/a_n|^k} = e^{-c|z|^k\sum_{|a_n|\le 2|z|}|a_n|^{-k}}$$

and

$$\sum_{|a_n|\le 2|z|} |a_n|^{-k} = \sum_{|a_n|\le 2|z|} |a_n|^{-s}|a_n|^{s-k} \le \sum_{|a_n|\le 2|z|} |a_n|^{-s}|2z|^{s-k} = C|z|^{s-k}.$$

The first term causes us to need to stay away from the zeros a_n assuming that z is not in any disc around an a_n of radius $|a_n|^{-k-1}$ we have

$$\prod_{|a_n|\le 2|z|} |1 - z/a_n| = \prod_{|a_n|\le 2|z|} |\frac{a_n - z}{a_n}| \ge \prod_{|a_n|\le 2|z|} |a_n|^{-k-2}$$

and

$$(k+2)\sum_{|a_n|\le 2|z|} \log|a_n| \le (k+2)\mathfrak{n}(2|z|)\log 2|z| \le c|z|^\rho \log 2|z| \le c|z|^s$$

if we take $\rho_0 < \rho < s$ and $|z| \ge 1$.

Corollary: There exists a sequence of radii $r_m \to \infty$ on which

$$|\prod E_k(z/a_n)| \ge e^{-c|z|^s} \quad \text{for} \quad |z| = r_m.$$

Proof: Take N suff large so that $\sum_{n\ge N}|a_n|^{-k-1} < 1/2$. Then between any two consecutive large integers L and $L+1$ there must exist a radius $L \le r \le L+1$ which does not intersect the union of the forbidden circles. Otherwise the union of the radii $[|a_n| - |a_n|^{-k-1}, |a_n| + |a_n|^{-k-1}]$ would cover $[L, L+1]$ which is not possible because the sum of their lengths is strictly smaller than $2 * 1/2$.

4. Analytic continuation

Given a holomorphic function f on Ω when can it be extended to be holomorphic on a larger set? What kind of uniqueness statements can we make about the extension? Think of the logarithm, extending "along" different curves can result in different values.

4.1. **Examples of analytic extension.**

In general the first question of *when can f be extended to be holomorphic on a larger set* is very difficult! It is better to use special structure in specific cases. For example recall the Schwarz reflection principal. Other examples would include analytic extension of ζ or Γ which rely on special formulae.

4.2. **Analytic continuation along a path.**

We start by re-analyzing our notion of function. Recall that every function f is really a triplet (f, Ω, R) where Ω is the domain and R is the range (for us these will still just be subsets of $\mathbb{C}$). Technically when we change either the domain or the range we are also changing the function. Of course it is often useful to realize that there are natural equivalence relations under some of these operations. For example we don't lose much by considering $(f, \Omega, f(\Omega))$, $(f, \Omega, \mathbb{C})$, and (f, Ω, R) for any $f(\Omega) \subset R \subset \mathbb{C}$ to be equivalent. On the other hand the dependence of the function on its domain is quite relevant in complex analysis.

Define: (g, Ω') extends (f, Ω) if $\Omega \subset \Omega'$ and $f = g$ on Ω.

Example: A holomorphic function (f, Ω) which has two distinct holomorphic extensions (g, Ω_1) and (h, Ω_2), i.e. Ω_1 and Ω_2 overlap outside of Ω and the extensions differ. (Natural example is logarithm extended around 0 in opposite directions).

Define: A *function element* is a pair (f, Ω) where Ω is a domain and f is a holomorphic function on Ω. For a given function element (f, Ω) define the *germ of f at $a \in \Omega$*, denoted $[f]_a$, to be the set of all function elements (g, D) such that $a \in D$ and $f(z) = g(z)$ in a neighborhood of a.

Notes: If (f, Ω) is a function element and $(g, D) \in [f]_a$ then $(f, \Omega) \in [g]_a$. So $[f]_a$ is an equivalence class of function elements, it is not a function element itself. The terminology "germ" is from botany, it is the "germ" from which something (more general than just a function) will grow. The germs $[f]_a$ and $[g]_b$ are not comparable when $a \neq b$.

Define: Let $\gamma : [0, 1] \to \mathbb{C}$ be a curve and suppose that for each $t \in [0, 1]$ there is a function element (f_t, Ω_t) so that

(1) $\gamma(t) \in \Omega_t$
(2) For each t there is $\delta > 0$ so that $|t - s| < \delta$ implies that $\gamma(s) \in \Omega_t$ (automatic from continuity / openness) *and*

$$[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$$

Then we say that (f_1, Ω_1) is the *analytic continuation* of (f_0, Ω_0) along the curve γ.

Again think of the logarithm, give examples.

Important content of (b) : $f_t(z) = f_s(z)$ whenever $|s-t| < \delta$ s.t. $\gamma(s) \in \Omega_t$ and z is in the connected component of $\Omega_t \cap \Omega_s$ containing $\gamma(s)$.

Showing the existence of an analytic continuation along a particular curve is *hard in general*, we use specific information about a particular function to do that. What we can make generalities about is *uniqueness criteria* of the analytic continuation if some exist (when are the analytic continuations along two different curves the same).

Proposition: Let γ be a path from a to b and (f_t, Ω_t) and (g_t, Δ_t) be two analytic continuations along γ so that $[f_0]_a = [g_0]_a$. Then $[f_1]_b = [g_1]_b$.

(Or, more simply, $f_1 = g_1$ in a neighborhood of b).

Proof: We will show that

$$T = \{t \in [0,1] : [f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}\}$$

is both open and closed in $[0,1]$. Note T is nonempty since $0 \in T$ so if we show T is clopen then $T = [0,1]$.

Open: Fix $t \in T \cap [0,1]$. There is $\delta > 0$ so that $|s-t| < \delta$ and $s \in [0,1]$ implies $\gamma(s) \in \Omega_t \cap \Delta_t$ and

$$[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)} \text{ and } [g_s]_{\gamma(s)} = [g_t]_{\gamma(s)}$$

Let U be the open subset of $\Omega_t \cap \Delta_t$ which contains $\gamma([s,t])$. Since $t \in T$ we know $f_t(z) = g_t(z)$ in a nbhd of $\gamma(t)$ and so also in U. Thus $[f_t]_{\gamma(s)} = [g_t]_{\gamma(s)}$ and so

$$[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)} = [g_t]_{\gamma(s)} = [g_t]_{\gamma(s)}.$$

Since $|s-t| < \delta$ was arbitrary we find T open.

Closed: Let t be a limit point of T, let $\delta > 0$ so that $|s-t| < \delta$ and $s \in [0,1]$ implies $\gamma(s) \in \Omega_t \cap \Delta_t$ and

$$[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)} \text{ and } [g_s]_{\gamma(s)} = [g_t]_{\gamma(s)}.$$

There is some $s \in T \cap (t-\delta, t+\delta)$ so taking this s let U be the connected component of $\gamma((t-\delta, t+\delta))$ in $\Omega_t \cap \Delta_t$. Since $\gamma(s) \in U$ and $f_s = g_s$ in a neighborhood of $\gamma(s)$ we have $f_s = g_s$ in U. But also $f_t = f_s$ and $g_t = g_s$ in U so $f_t = g_t$ in U and in particular in a neighborhood of $\gamma(t)$.

Definition: If γ is a path from a to b and (f_t, Ω_t) is an analytic continuation on γ then the germ $[f_1]_b$ is defined to be the analytic continuation of $[f_0]_a$ along γ.

By the previous proposition any two function elements which continue a function element from $[f_0]_a$ along γ must agree in a neighborhood of b meaning that they are in the same germ $[f_1]_b$. Also note that the choice of domains Ω_t in the continuation does not change the value of $[f_1]_b$. These make the above definition unambiguous.

Definition: If (f, Ω) is a function element then *the complete analytic function obtained from* (f, Ω) is the collection $\mathcal{F}$ of all germs $[g]_b$ for which there is a point $a \in \Omega$ and a path γ from a to b so that $[g]_b$ is the continuation of $[f]_a$ along γ.

(Note: the point a is not important because Ω is connected so given a continuation from a to b there is also one from any other point of Ω to b via concatenation.)

Example: Create a complete analytic function out of $\log(z)$ defined initially in a neighborhood of 1.

4.3. **Monodromy theorem.** What if we continue $[f_0]_a = [g_0]_a$ along two *different* curves γ and η. In general, of course, $[f_1]_b$ and $[g_1]_b$ are not the same. Think of non-homotopic paths around zero for the logarithm.

Intuitively homotopy becomes a relevant criterion.

Definition: Given a function element (f, Δ) and a domain $\Omega \supset \Delta$ say that (f, Δ) *admits unrestricted continuation on* Ω if for any path γ in Ω with initial point in Δ there is an analytic continuation of f along γ.

Theorem: Let (f, Δ) be a function element and $\Omega \supset \Delta$ a region on which (f, Δ) admits unrestricted analytic continuation. Let $a \in \Delta$, $b \in \Omega$ and γ, η be paths in Ω from a to b and $[f_1]_b$ and $[g_1]_b$ be analytic continuations of $[f_0]_a$ along γ and η respectively. If γ_0 and γ_1 are homotopic in Ω then $[f_1]_b = [g_1]_b$.

This will be an open/closed argument using the homotopy to continuously move one analytic continuation to the other. Similar to when we proved the homotopy version of Cauchy's theorem. Thus the main important technical point will be to show *the analytic continuation along two nearby curves gives the same germ at the end point.*

In this direction we start by analyzing the radius of convergence of the power series expansion along a continuation.

Lemma: Let $\gamma : [0, 1] \to \mathbb{C}$ be a path and (f_t, Δ_t) be an analytic continuation along γ. Let $R(t)$ be the radius of convergence of the power series expansion of f_t about $z_t = \gamma(t)$. Either $R(t) \equiv +\infty$ or $R(t)$ is continuous.

Corollary: $R(t)$ is bounded from below on $[0, 1]$.

Proof: If $R(t) = +\infty$ for some t then f_t can be extended to be an entire function g. In that case $f_s(z) = f_t(z)$ in D_s so f_s can be extended to be entire as well and $R(s) = +\infty$. (Apply previous lemma since $(g, \mathbb{C})$ is another continuation of f_t along the same curve).

Otherwise $R(t) < +\infty$ for all $t \in [0, 1]$. Fixing a t let

$$f_t(z) = \sum a_n(t)(z - \gamma(t))^n$$

be the power series expansion of f_t about $\gamma(t)$, note that f_t can be extended to be analytic on $B(\gamma(t), R(t))$ if it was not already. Let $\delta > 0$ sufficiently small so that $|s-t| < \delta$ implies $\gamma(s) \in \Delta_t \cap D(\gamma(t), R(t))$ and, by the analytic continuation, $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$. Also f_s can be extended to $D(\gamma(s), R(s))$ and $f_t \equiv f_s$ in $D(\gamma(s), R(s)) \cap D(\gamma(t), R(t))$. This means that f_s can be extended analytically to $D(\gamma(s), R(s)) \cup D(\gamma(t), R(t))$ which means that

$$R(s) \geq \inf\{|z - w| : w \in \partial D(\gamma(t), R(t))\} \geq R(t) - |\gamma(s) - \gamma(t)|.$$

Reversing the roles of s and t gives the other inequality

$$|R(t) - R(s)| \leq |\gamma(t) - \gamma(s)|$$

for $|t - s| \leq \delta$ (and then actually for all $t, s \in [0, 1]$).

Lemma: Let γ be a path from a to b and (f_t, Ω_t) be an analytic continuation along γ. There is $\varepsilon > 0$ so that if η is another path from a to b with $\sup_t |\eta(t) - \gamma(t)| \leq \varepsilon$ and (g_t, Δ_t) is a continuation on η with $[g_0]_a = [f_0]_a$ then $[g_1]_b = [f_1]_b$.

Proof. Let $R(t)$ be the radius of convergence for f_t about $\gamma(t)$ let $\varepsilon < \frac{1}{2} \min R(t)$ and (g_t, Δ_t) and η as in the lemma. Without loss we can assume that $\Omega_t = D_{R(t)}(\gamma(t))$ and Δ_t are disks as well.

Since $|\eta(t) - \gamma(t)| < \varepsilon < R(t)/2$ then $\sigma(t) \in \Omega_t \cap \Delta_t$ for all $t > 0$ and so it makes sense to compare f_t and g_t on $\Omega_t \cap \Delta_t$ (nontrivial intersection).

Define

$$T = \{t \in [0, 1] : f_t = g_t \text{ in } \Omega_t \cap \Delta_t\}.$$

and we want to show $1 \in T$. We know $0 \in T$ by assumption of the Lemma. So we show T is open and closed and this gives the result.

Open: Fix t and let $\delta > 0$ sufficiently small so that

$$|\gamma(t) - \gamma(s)| < \varepsilon, \ [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$$

and

$$|\eta(t) - \eta(s)| < \varepsilon, \ [g_s]_{\eta(s)} = [g_t]_{\eta(s)}, \text{ and } \eta(s) \in \Delta_t$$

for $|t - s| < \delta$. Then we show $\Omega_t \cap \Omega_s \cap \Delta_s \cap \Delta_t \neq \emptyset$ for $|t - s| < \delta$, actually $\eta(s)$ is in the intersection.

$$|\eta(s) - \gamma(s)| < \varepsilon < R(s)$$

so $\eta(s) \in \Omega_s$

$$|\eta(s) - \gamma(t)| \leq |\eta(s) - \gamma(s)| + |\gamma(s) - \gamma(t)| < 2\varepsilon < R(t)$$

so $\eta(s) \in \Omega_t$.

Thus

$$\eta(s) \in \Omega_t \cap \Omega_s \cap \Delta_s \cap \Delta_t = U$$

and since $t \in T$ we know $f_t \equiv g_t$ in $\Omega_t \cap \Delta_t$ which is a superset of U. On the other hand $f_s \equiv f_t$ on $\Omega_s \cap \Omega_t$ and $g_s \equiv g_t$ on $\Delta_s \cap \Delta_t$. All supersets of U so

$$g_s \equiv g_t \equiv f_t \equiv f_s \text{ on } U.$$

By unique continuation $g_s \equiv f_s$ in all of $\Omega_s \cap \Delta_s$. Thus $(t - \delta, t + \delta) \subset T$. Closed proof is quite similar.

Proof of monodromy theorem: Since γ_0 and γ_1 are FEP homotopic in Ω there is $\Gamma(t, u) : [0, 1] \times [0, 1] \to \Omega$ homotopy between them (list properties). Fix $u \in [0, 1]$ then $\Gamma(t, u)$ is a path from a to b in Ω so there is an analytic continuation

$$(h_{t,u}, \Delta_{t,u}) \text{ for } t \in [0, 1].$$

Then $[f_1]_b = [h_{1,0}]_b$ and $[g_1]_b = [h_{1,1}]_b$ so it suffices to show

$$[h_{1,u}]_b = [h_{1,0}]_b \text{ for all } u \in [0,1]$$

Call

$$U = \{u \in [0,1] : [h_{1,u}]_b = [h_{1,0}]_b\}$$

we show U is open and closed in $[0,1]$.

Claim: For each $u \in [0,1]$ there is $\delta > 0$ s.t. if $|v-u| < \delta$ then $[h_{1,v}]_b = [h_{1,u}]_b$.

Apply previous lemma to find $\varepsilon > 0$ so that if η is any path from a to b with $|\Gamma(t,u) - \eta(t)| < \varepsilon$ for all $t \in [0,1]$ and if (k_t, Λ_t) is an analytic continuation on η then

$$[h_{1,u}]_b = [k_1]_b.$$

Sicne Γ is uniformly continuous there is $\delta > 0$ so that

$$\sup_t |\Gamma(t,u) - \Gamma(t,v)| < \varepsilon$$

for $|u - v| < \varepsilon$.

The claim implies both U open and U closed.

$\square$

Corollary: If (f, Δ) admits unrestricted continuation on simply connected Ω then there is a (unique) holomorphic $F : \Omega \to \mathbb{C}$ with $f(z) = F(z)$ on Δ.

Proof: Fix $z_0 \in \Delta$, given $z \in \Omega$ there is a path from z_0 to z call γ such a path. Define $F(z,\gamma)$ to be the analytic continuation of f along γ. $F(z,\gamma) = F(z,\eta)$ for any other path η from z_0 to z because γ and η are homotopic. Then just define $F(z) = F(z,\gamma)$ for any particular choice γ from z_0 to z. Then let's show F is holomorphic is a neighborhood of z, take γ and the analytic continuation along γ from the definition of $F(z)$. Sufficiently close point $|w - z|$ concatenate γ with the line segment from z to w and $F(w)$ is the continuation along that curve, but we can also make explicit continuation by concatenating with (f_1, Δ_1) on the last small segment.

5. Elliptic functions

In this section we consider doubly periodic (meromorphic) functions on $\mathbb{C}$. That is there is a pair ω_1 and ω_2 nonzero complex numbers so that

$$f(z + \omega_j) = f(z) \text{ for } j = 1, 2.$$

If $\omega_2/\omega_1 \in \mathbb{R}$ (i.e. linearly dependent periods in $\mathbb{R}^2$) then the case is not interesting, either f is periodic with a real period (when ω_2/ω_1 is rational) or constant (when ω_2/ω_1 is irrational).

Let $\tau = \omega_2/\omega_1$, we can assume that $\mathrm{Im}(\tau) > 0$. We can rescale to

$$F(z) = f(\omega_1 z)$$

which has periods 1 and τ if and only if f has periods ω_1 and ω_2. We can thus normalize and assume f has periods 1 and τ with $\mathrm{Im}(\tau) > 0$.

Now f is periodic with respect to

$$f(z+n+m\tau)=f(z)$$

for $n,m\in\mathbb{Z}$ so the lattice

$$\Lambda=\{n+m\tau:(n,m)\in\mathbb{Z}^2\}$$

is called the periodicity lattice of f generated by the periods $(1,\tau)$.

The fundamental parallelogram of the lattice is

$$P_0=\{z\in\mathbb{C}:z=a+b\tau:(a,b)\in[0,1)^2\}$$

(draw picture). The lattice translations of P_0 tile $\mathbb{C}$.

Saw that $z\tilde{w}$ congruent modulo Λ if

$$z=w+n+\tau m \quad \text{for some} \quad n,m\in\mathbb{Z}.$$

i.e. $z-w\in\Lambda$. Of course $z\tilde{w}$ implies $f(z)=f(w)$. Thus f is determined by its values on P_0.

One can also choose any translation period parallelogram P_0+h for $h\in\mathbb{C}$ and f is also determined by its values on P_h (every point in $\mathbb{C}$ is congruent to a unique point in P_0+h).

Lemma: An entire doubly period function is constant.

f is bounded on P_0 since its closure is compact, and so f is bounded on $\mathbb{C}$ implying, by Liouville, that f is constant.

Thus our interest centers on meromorphic functions. A non-constant doubly periodic meromorphic function is called an *elliptic function*.

Elliptic functions can only have finitely many zeros and poles in P_0

Theorem: The total number of poles of an elliptic function in P_0 is always ≥ 2.

Proof: Suppose first f has no poles on ∂P_0, then residue theorem implies

$$\int_{\partial P_0} f(z)\ dz=2\pi i\sum \operatorname{Res}(f,z_i)$$

where z_i are the poles. The claim is that the integral is zero meaning there must be at least two poles in P_0 with multiplicity (a simple pole cannot have zero residue).

$$\int_{\partial P_0} f(z)\ dz=\int_0^1 f(z)\ dz+\int_1^{1+\tau} f(z)\ dz+\int_{1+\tau}^{\tau} f(z)\ dz+\int_{\tau}^{0} f(z)\ dz$$

integrals on opposite sides cancel e.g.

$$\begin{aligned}\int_0^1 f(z)\ dz+\int_{1+\tau}^{\tau} f(z)\ dz&=\int_0^1 f(z)dz+\int_1^0 f(s+\tau)ds\\&=\int_0^1 f(z)dz+\int_1^0 f(s)ds=0.\end{aligned}$$

If f has a pole on ∂P_0 choose $h>0$ small so that P_0+h has no poles on it's boundary (possible b/c there are only finitely many poles in any compact region).

The total number of poles counted with multiplicity is called the **order** of the elliptic function.

Theorem: An elliptic function of order m has m zeros in P_0.

proof: Assume no zeros or poles on ∂P_0 (again achievable by a shift to $P_0 + h$ if necessary). Then argument principle

$$\int_{\partial P_0} \frac{f'(z)}{f(z)}\ dz = 2\pi i(N_z - N_p).$$

As before the boundary integral is zero by the periodicity, line integrals on opposite faces cancel.

The same argument shows that the equation $f(z) = c$ has the same number of solutions in P_0 as the order for any $c \in \mathbb{C}$.

5.1. **Existence.** Are there any elliptic functions?

Start with a single period idea

$$F(z) = \sum_n \frac{1}{z+n}$$

this is not absolutely summable, which can be fixed by (DO BELOW IDEA INSTEAD) summing symmetrically

$$F(z) = \sum_{|n|\le N} \frac{1}{z+n} = \frac{1}{z} + \sum_1^\infty \left[\frac{1}{z+n} + \frac{1}{z-n}\right]$$

the last sum is absolutely convergent since the terms are $\frac{2z}{z^2-n^2}$ which is absolutely summable for all z. Thus F is meromorphic with poles at the integers, it turns out it is $F(z) = \pi\cot(\pi z)$.

(Can also add and subtract $\frac{1}{n}$ for each n resulting in an absolutely convergent series).

Try a similar idea on a lattice $\Lambda \subset \mathbb{C}$

$$\sum_{\omega\in\Lambda} \frac{1}{(z+\omega)^2}$$

again this is just barely not absolutely summable. If it were summable it would be Λ-periodic.

Call $\Lambda^* = \Lambda \setminus \{0\}$ and consider

$$\frac{1}{z^2} + \sum_{\omega\in\Lambda^*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]$$

the terms are

$$\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{-z^2 - 2z\omega}{(z+\omega)^2\omega^2} = O(\frac{1}{\omega^3}) \ \text{ as } \ |\omega| \to \infty.$$

Lemma: For $r > 2$

$$\sum_{n,m\in\mathbb{Z}^2} \frac{1}{(|n|+|m|)^r} < +\infty$$

and

$$\sum_{\Lambda^*} \frac{1}{|n+\tau m|^r} < +\infty.$$

Proof: Double sum for the first one, doing integral comparison.

For the second want to show

$$|n| + |m| \leq C(\tau)|n+\tau m|$$

for an appropriate $C(\tau) > 1$.

With $\tau = s + it$ and $t > 0$

$$|n+m\tau| = \sqrt{(n+ms)^2 + (mt)^2} \sim |n+ms| + |mt| \sim |n+ms| + |m|$$

Show this with the matrix norm of the inverse of $(n,m) \mapsto (n+sm, tm)$.

Final claim is that

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]$$

is a meromorphic function on $\mathbb{C}$ with periods $\omega \in \Lambda$ and double poles at each lattice point.

The issue of poles: split sum into $|\omega| \leq 2R$ and $|\omega| > 2R$. The outer sum converges to zero uniformly, the inner sum has the correct double poles at all lattice points. The order of poles is preserved in the limit by argument principle.

The issue of Λ-periodicity: The derivative can be computed by term by term differentiation

$$\mathcal{P}'(z) = -2\sum_{\omega\in\Lambda} \frac{1}{(z+\omega)^3}$$

this sum is dominated by an absolutely convergent sum whenever $z \notin \Lambda$, the series is obviously Λ-periodic now. This means

$$\mathcal{P}(z+1) = \mathcal{P}(z) + a \text{ and } \mathcal{P}(z+\tau) = \mathcal{P}(z) + b$$

for some $a, b \in \mathbb{C}$ and all $z \in \mathbb{C}$. However $\mathcal{P}$ is also an even function (simply plug in the definition and note $-\Lambda = \Lambda$) so

$$\mathcal{P}(-1/2) = \mathcal{P}(1/2) \text{ and } \mathcal{P}(\tau/2) = \mathcal{P}(-\tau/2)$$

so

$$\mathcal{P}(1/2) = \mathcal{P}(-1/2) + a = \mathcal{P}(1/2) + a$$

so $a = 0$ and similarly $b = 0$.

Thus $\mathcal{P}$ as defined above is a doubly periodic meromorphic function with double poles at each lattice point of Λ. This is our first example of an elliptic function. Note that $\mathcal{P}'(z)$ is also an elliptic function, it has order 3 since its poles in P_0 are exactly the pole of order 3 at the origin.

Next note that $\mathcal{P}'(z)$ is odd so a similar argument to above shows that

$$\mathcal{P}'(1/2) = \mathcal{P}'(\tau/2) = \mathcal{P}'(\frac{1+\tau}{2}) = 0.$$

Because $\mathcal{P}'$ is an elliptic function of order 3 these are *exactly* the zeros of $\mathcal{P}'$ in the fundamental parallelogram.

Define

$$\mathcal{P}(1/2) = e_1, \ \mathcal{P}(\tau/2) = e_2, \quad \text{and} \ \ \mathcal{P}(\frac{1+\tau}{2}) = e_3$$

then $\mathcal{P}(z) = e_1$ has a double root at $1/2$ and since $\mathcal{P}$ is order 2 these are all the zeros of $\mathcal{P}(z) - e_1$. Similar argument for e_2 and e_3. In particular these three values must be distinct otherwise $\mathcal{P}(z) - e_j$ would have more than two roots in P_0 which is not allowed.

This leads to the following *differential equation* for $\mathcal{P}$

Lemma:

$$\mathcal{P}'^2 = 4(\mathcal{P} - e_1)(\mathcal{P} - e_2)(\mathcal{P} - e_3).$$

Proof: The function $F(z) = (\mathcal{P} - e_1)(\mathcal{P} - e_2)(\mathcal{P} - e_3)$ has double roots exactly at $1/2$, $\tau/2$ and $(1+\tau)/2$, those are also exactly the roots of $(\mathcal{P}')^2$. F has poles of order 6 at lattice points and so does $(\mathcal{P}')^2$. Thus the ratio $F/(\mathcal{P}')^2$ is has removable singularities at those locations and is holomorphic otherwise, so by Liouville it is constant. To find the value of the constant we compute the highest order term in the Laurent expansions at 0

$$\mathcal{P}(z) = \frac{1}{z^2} + \cdots \quad \text{and} \ \ \mathcal{P}'(z) = -\frac{2}{z^3} + \cdots$$

so

$$F(z) = \frac{1}{z^6} + \cdots$$

near the origin which leads to the constant 4 above.

It turns out that $\mathcal{P}$ generates all the elliptic functions with period lattice Λ in a simple way

Theorem: Every elliptic function with period lattice Λ is a rational function of $\mathcal{P}$ and $\mathcal{P}'$.

First we show this for *even* elliptic functions F.

If F has a pole or zero at the origin it must be of even order. Thus $F\mathcal{P}^m$ has no zero or pole at the lattice points for an appropriate choice of m integer, so WLOG we can just assume F has no poles or zeros on Λ.

Now we construct a doubly periodic function based on $\mathcal{P}$ with the same zeros and poles as F.

Note that $\mathcal{P}(z) - \mathcal{P}(a)$ has exactly a double zero at a if a is a half-period, otherwise there are two distinct zeros at a and $-a$.

Similarly for F if a is a zero then so is $-a$, and $-a$ is congruent to a only when a is a half-period in which case the zero must be of even order (CHECK). Thus if $a_j, -a_j$ are the zeros of F counted with multiplicity then

$$\prod_{j=1}^{m} [\mathcal{P}(z) - \mathcal{P}(a_j)]$$

has the exact same roots as F. A similar argument with the poles $b_1, -b_1, \ldots, b_m, -b_m$ shows that

$$G(z) = \frac{\prod_{j=1}^m [\mathcal{P}(z) - \mathcal{P}(a_j)]}{\prod_{j=1}^m [\mathcal{P}(z) - \mathcal{P}(b_j)]}$$

has the same zeros and poles as F. Thus F/G is entire and hence constant.

Next for a general F elliptic we can write

$$F = \frac{F(z) + F(-z)}{2} + \frac{F(z) - F(-z)}{2} = F_{even}(z) + F_{odd}(z)$$

which is an even-odd decomposition. Then

$$F_{odd}(z)/\mathcal{P}'(z)$$

is an even elliptic function so the previous argument applies.

5.2. **The modular function.** Recall the differential equation

$$\mathcal{P}'^2 = 4(\mathcal{P} - e_1)(\mathcal{P} - e_2)(\mathcal{P} - e_3).$$

with

$$\mathcal{P}(1/2) = e_1, \ \mathcal{P}(\tau/2) = e_2, \ \text{ and } \ \mathcal{P}(\frac{1+\tau}{2}) = e_3.$$

Now we regard these as functions of τ and we define the modular fuction

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}$$

The analyticity of e_j in $\mathbb{H}$ is clear from the sum formula

$$e_j(\tau) = \mathcal{P}_\tau(z_j) = \frac{1}{z_j^2} + \sum_{(n,m)\neq(0,0)} \left[\frac{1}{(z_j + n + m\tau)^2} - \frac{1}{n + m\tau} \right]$$

the series converges uniformly in compact subsets of $\mathcal{H}$ by previous arguments.

Since the factors e_j are all distinct the function λ is analytic in $\mathbb{H}$ and does not take the values $0, 1, \infty$.

Consider the period transformation

$$\omega_1' = a\omega_1 + b\omega_2$$
$$\omega_2' = c\omega_1 + d\omega_2$$

with unimodular matrix M (i.e. integer entries and determinant ± 1). The unimodular matrices form a group $\mathrm{GL}_2(\mathbb{Z})$. By invertibility of the transformation the lattices Λ and Λ' are the same. Thus the $\mathcal{P}$ functions are invariant. It is possible though that the values e_1, e_2, e_3 are permuted (look at the differential equation).

In order for $\omega_j'/2 \sim \omega_j/2$ (modulo lattice) for $j = 1, 2$ we should have $a, d = 1 \bmod 2$ and $b, c = 0 \bmod 2$. This leads to invariance

$$\lambda\left(\frac{a\tau + b}{c\tau + d}\right) = \lambda(\tau) \ \text{ when } \ M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod 2.$$

The unimodular matrices satisfying this property form a subgroup call the congruence subgroup modulo 2. Recall that this is also a subgroup of the FLTs which preserve the upper half-space.

Functional equations

$$\lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau)-1}, \quad \text{and} \quad \lambda(-\frac{1}{\tau}) = 1 - \lambda(\tau).$$

The modular function λ behaves nicely as a conformal mapping. Call $\Omega = \{0 < \mathrm{Re}(z) < 1\} \cap \mathbb{H} \cap D_{1/2}(1/2)^c$ and Ω' the reflection of Ω across the imaginary axis, and $\tilde{\Omega}$ the interior of the union of the closures of Ω and Ω'

Theorem 1. *λ maps Ω conformally to $\mathbb{H}$, and maps Ω' conformally to the lower half plane, and maps $\tilde{\Omega}$ conformally to $\mathbb{C} \setminus \{0,1\}$.*

Theorem 2. *Every point $\tau \in \mathbb{H}$ is equivalent to exactly one point of $\tilde{\Omega}$ under the congruence subgroup modulo 2.*

Thus λ is a locally conformal covering of $\mathbb{C} \setminus \{0,1\}$.

6. Picard theorems

Theorem 3. *(Little Picard) If f is entire and misses two points the f is constant.*

Proof sketch: First if f misses a and b distinct we can transform to

$$\frac{f(z)-a}{b-a}$$

which misses 0 and 1.

Now WLOG suppose f misses 0 and 1. Lift f to an entire mapping $\hat{f} : \mathbb{C} \to \mathbb{H}$ via the locally conformal covering map $\lambda : \mathbb{H} \to \mathbb{C} \setminus \{0,1\}$ i.e. there is a holomorphic map $\hat{f} : \mathbb{C} \to \mathbb{H}$ with

$$\lambda(\hat{f}(z)) = f(z).$$

This requires the monodromy theorem and is related to homework 5 problems, we will revisit it later. Then Liouville and the conformal mapping $\mathbb{H} \to \mathbb{D}$ shows $\hat{f}$ is constant which also implies that f is constant.

Theorem 4. *(Big Picard) If f has an essential singularity at 0 then in every punctured neighborhood of 0 f takes all values in $\mathbb{C}$ infinitely often with at most one exception.*

The idea is similar to little Picard, now we "zoom in" on the essential singularity to create a contradiction. We need a refined compactness result which is a natural extension of Montel's theorem (on normal families) to the setting suggested by Picard's theorem.

Recall definition of normal family for holomorphic mappings, recall Montel's theorem: family $\mathcal{F}$ is uniformly bounded on compact subsets of Ω then $\mathcal{F}$ is a normal family.

Say that $f_n \to \infty$ uniformly on compact subsets of Ω if $1/f_n$ converges to 0 uniformly on compact subsets of Ω.

We say that a family $\mathcal{F}$ of holomorphic functions on Ω is normal in the extended sense if for any sequence $f_n \in \mathcal{F}$ there is a either a subsequence converging uniformly on compact subsets of Ω or a subsequence converging uniformly to ∞ on compact subsets of Ω.

Lemma 1. *If a family $\mathcal{F}$ of holomorphic functions on a domain U takes values in $\mathbb{H}$ then $\mathcal{F}$ is normal in the extended sense.*

Proof. Take $\varphi = \frac{z-i}{z+i}$ conformal map from $\mathbb{H}$ to $\mathbb{D}$. Call

$$\mathcal{G} = \{\varphi \circ f : f \in \mathcal{F}\}.$$

This family is uniformly bounded so it is normal. If $\varphi \circ f_n$ converges uniformly on compact subsets of $\mathbb{D}$ to some g holomorphic on $\mathbb{D}$ then $g(\mathbb{D}) \subset \overline{\mathbb{D}}$. If g is non-constant then g is an open mapping so $g(\mathbb{D})$ is open and so $g(\mathbb{D}) \subset \mathbb{D}$. Then in that case

$$f_n = \varphi^{-1} \circ \varphi \circ g_n$$

converges locally uniformly on $\mathbb{H}$ to $\varphi^{-1} \circ g$. If $g \equiv \zeta$ is constant and $\zeta \in \mathbb{D}$ the same argument applies. If $\zeta \in \partial\mathbb{D} \setminus \{1\}$ then φ^{-1} is continuous up to ζ so again same result as before. Finally if $\zeta = \{1\}$ then

$$|\varphi^{-1}(w)| \to \infty \quad \text{as} \quad w \to \zeta = 1$$

i.e. $\varphi^{-1}(w)^{-1} \to 0$ as $w \to 1$ so we get the case

$$f_n = \varphi^{-1} \circ \varphi \circ g_n \to \infty$$

uniformly on compact subsets of $\mathbb{H}$. □

Lemma 2. *Let $\mathcal{F}$ be a family of holomorphic functions on a domain Ω. $\mathcal{F}$ is normal if and only if for each $z \in \Omega$ there is U open containing z so that $\mathcal{F}|U = \{f|_U : f \in \mathcal{F}\}$ is normal.*

Proof. Let $K_n \subset K_{n+1} \subset \cdots$ an exhaustion of Ω by compact sets. Each z has a neighborhood U_z on which $\mathcal{F}|U_z$ is normal. Then $(U_z)_{z \in K_j}$ covers K_j take a finite subcover $(U_{z_{j,n}})_{j=1}^{J_n}$ of K_j . Given a sequence $f_n \in \mathcal{F}$ by a subsequence diagonalization argument we can find a subsequence converging locally uniformly on compact subsets of $\cup_{j,n} U_j = \Omega$. □

Theorem 5 (Montel)**.** *Suppose $\mathcal{F}$ is a family of holomorphic functions on a domain Ω so that $f(\Omega) \subset \mathbb{C} \setminus \{a, b\}$ for two distinct a, b, then $\mathcal{F}$ is a normal family (in the extended sense).*

We can show that normality is a local property so it will suffice to prove the theorem for $\Omega = \mathbb{D}$.

Proof. By a FLT we can assume that $\{a, b\} = \{0, 1\}$. We can always take a subsequence so that $f_n(0)$ converges (possibly to ∞). Let us first remove the cases when $f_n(0) \to 0$ or $f_n(0) \to 1$. In the second case f_n are all nonvanishing so they each have a holomorphic square root $h_n(z)^2 = f_n(z)$ choose the branch so that $h_n(0) \to -1$. Then h_n are holomorphic and miss $\{0,1\}$ and $h_n(0) \not\to -1 \not\in \{0,1\}$. Then if h_n are normal f_n are as well since $f_n = h_n^2$. A similar argument works if $f_n(0) \to 0$ (take $h_n(z)^2 = 1 - f_n(z)$).

Since $\mathbb{D}$ is simply connected we can lift each f_n to $\hat{f}_n : \mathbb{D} \to \mathbb{H}$ via the λ covering map $\lambda(\hat{f}_n) = f_n$. We can also make a normalization that

$$\hat{f}_n(0) \in \tilde{\Omega}$$

the fundamental domain of the congruence subgroup modulo 2 equivalence relation.

The family $\hat{f}_n$ is normal in the extended sense by the previous Lemma. So up to a subsequence $\hat{f}_n$ converges uniformly on compact subsets of $\mathbb{D}$ to some $\hat{f}$ or to ∞. In the second case

$$f_n = \lambda(\hat{f}_n) \to \infty \text{ on compact subsets of } \mathbb{D}$$

so we are done.

In the first case, where $\hat{f}_n \to \hat{f}$ on compact subsets of $\mathbb{D}$, note that the range of $\hat{f}$ is contained in the closed half-plane $\overline{\mathbb{H}}$. If $\hat{f}$ is non-constant then it is an open mapping so the range of $\hat{f}$ is open so $\hat{f}(\mathbb{H}) \subset \mathbb{H}$. In this case

$$f_n = \lambda(\varphi(\hat{f}_n))$$

converges uniformly on compact sets.

Otherwise $\hat{f} \in \partial\mathbb{H}$ is constant $\zeta \in \partial\mathbb{H}$, in particular $\hat{f}_n(0) \to \zeta$. However $\hat{f}_n(0)$ is in the fundamental domain $\tilde{\Omega}$ so the only possibilities are $\zeta \in \{-1, 0, 1\}$. So either $f_n(0) \to 0$ or $f_n(0) \to 1$. But we have already normalized that neither of these cases occur so we are done.

□

Theorem 6. *(Big Picard) If f has an essential singularity at 0 then in every punctured neighborhood of 0 f takes all values in $\mathbb{C}$ infinitely often with at most one exception.*

Proof. Suppose that two values are only taken finitely often by f in some punctured disk $D_r(0) \setminus \{0\}$, without loss by a linear transformation these values can be taken to be $\{0,1\}$. By dilation we can assume that f is holomorphic in $\mathbb{D} \setminus \{0\}$ and $\{0,1\}$ are missed by f in $\mathbb{D} \setminus \{0\}$.

Consider the family

$$\mathcal{F} = \{f(rz) | 0 < r < 1\}$$

of holomorphic functions on the punctured disk $\mathbb{D}\setminus\{0\}$. By Montel's theorem this is an extended normal family. If

$$\lim_{|z|\to 0} |f(z)| = \infty$$

then f has a pole and 0 is not an essential singularity. Thus there is a sequence of points $z_n \to 0$ so that $|z_n|$ is decreasing and $|f(z_n)|$ is a bounded sequence. Call $r_n = |z_n|$. Then, up to taking a subsequence, $f_n(z) = f(r_n z)$

converges locally uniformly on $\mathbb{C} \setminus \{0\}$ to some g holomorphic on $\mathbb{C} \setminus \{0\}$. Since g is holomorphic $\sup_{|z|=1} g < +\infty$ so

$$M = \sup_n \sup_{|z|=r_n} |f(z)| < +\infty$$

since $\sup_{|z|=r_n} |f(z)| \to \sup_{|z|=1} |g(z)|$. However now we can apply the maximum modulus principle in $D_{r_n} \setminus D_{r_{n+1}}$ to find

$$\sup_{D_{r_1} \setminus \{0\}} |f(z)| \leq M$$

which implies that 0 is a removable singularity for f and this is a contradiction.

$\square$

7. Review

FINAL EXAM: Monday, May 2, 1-3pm

- Holomorphic functions
- Cauchy-Riemann equations
- Goursat Theorem
- Principals
- Cauchy's Theorem (homotopic curves)
- Cauchy's integral formula
- Power series expansion / holomorphic functions are analytic
- Liouville's theorem
- Runge's Theorem
- isolated singularities
- Laurent series
- residue theorem
- applications to compute definite integrals
- Rouche's Theorem
- Maximum modulus principle, holomorphic functions are open mappings
- Conformal mappings
- Schwartz lemma
- Fractional linear transformations
- disk / upper half plane automorphisms
- Normal families, Montel's theorem
- Riemann mapping theorem
- Infinite products
- Weierstrass factorization theorem
- Hadamard factorization theorem
- Analytic continuation
- monodromy
- Elliptic functions
- Montel-Caratheodory theorem

- Picard's theorem.