Chapter 17
Video Poker

Video meliora, deteriora sequor, as we said at college.

William Makepeace Thackeray, *The Virginians*

Video poker is an electronic form of five-card draw poker that dates back to the late 1970s. The player is dealt five cards and is allowed to replace any number of them by an equal number of cards drawn from the unseen deck. The rank of the resulting hand (and the bet size) determines the amount paid out to the player. We focus on two specific video poker games, Jacks or Better (Section 17.1), the version closest to five-card draw, and Deuces Wild (Section 17.2), a version with the additional complication of four wild cards. With optimal play, the Deuces Wild player has a slight advantage over the house, while the Jacks or Better player has a slight disadvantage.

17.1 Jacks or Better

Video poker is played on a machine that resembles a slot machine with a video monitor. Typically, the player inserts from one to five coins (currency and tickets are also accepted) to place a bet. He then receives five cards face up on the screen, with each of the $\binom{52}{5} = 2{,}598{,}960$ possible hands equally likely. (The order of the five cards is irrelevant.) For each card, the player must then decide whether to hold or discard that card. Thus, there are $2^5 = 32$ ways to play the hand. If he discards k cards, he is dealt k new cards, with each of the $\binom{47}{k}$ possibilities equally likely. The player then receives his payout, which depends on the amount he bet and the rank of his final hand.

The payout schedule for full-pay *Jacks or Better* is shown in Table 17.1. Unlike in Example 1.1.10 on p. 6, here we distinguish between a royal flush and a (nonroyal) straight flush. We use the term "full-pay" to emphasize the

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fact that there are similar machines with less favorable payout schedules. It should also be mentioned that, typically, to qualify for the 800 for 1 payout on a royal flush, the player must bet five coins. We assume that the player does this, and we define the total value of these five coins to be one unit.

Table 17.1 The full-pay Jacks or Better payoff odds, assuming a maximum-coin bet, and the pre-draw frequencies.

rank	payoff odds	number of ways
royal flush	800 for 1	4
straight flush	50 for 1	36
four of a kind	25 for 1	624
full house	9 for 1	3,744
flush	6 for 1	5,108
straight	4 for 1	10,200
three of a kind	3 for 1	54,912
two pair	2 for 1	123,552
pair of jacks or better	1 for 1	337,920
other	0 for 1	2,062,860
total		2,598,960

The primary issue in Jacks or Better is to determine, given the player's initial five-card hand, which cards should be held. Let us consider an example. Suppose that the player is dealt A♣-Q♢-J♣-T♣-9♣ (in any order, of course). There are several plausible strategies.

If the player holds the four-card one-gap straight A♣-Q♢-J♣-T♣, his payout R from a one-unit bet has conditional expected value

$$\mathrm{E}[R] = 4\left(\frac{4}{47}\right) + 1\left(\frac{9}{47}\right) + 0\left(\frac{34}{47}\right) = \frac{25}{47} \approx 0.531915. \tag{17.1}$$

For convenience we do not write out the conditioning event, namely "player is dealt A♣-Q♢-J♣-T♣-9♣ and holds A♣-Q♢-J♣-T♣."

If the player holds the four-card open-ended straight Q♢-J♣-T♣-9♣,

$$\mathrm{E}[R] = 4\left(\frac{8}{47}\right) + 1\left(\frac{6}{47}\right) + 0\left(\frac{33}{47}\right) = \frac{38}{47} \approx 0.808511. \tag{17.2}$$

If the player holds the four-card flush A♣-J♣-T♣-9♣,

$$\mathrm{E}[R] = 6\left(\frac{9}{47}\right) + 1\left(\frac{6}{47}\right) + 0\left(\frac{32}{47}\right) = \frac{60}{47} \approx 1.276596. \tag{17.3}$$

If the player holds the three-card open-ended straight flush J♣-T♣-9♣,

$$\begin{aligned}
\mathrm{E}[R] = 50\,&\frac{3\binom{2}{2}\binom{45}{0}}{\binom{47}{2}} + 6\,\frac{\left[\binom{9}{2} - 3\binom{2}{2}\binom{7}{0}\right]\binom{38}{0}}{\binom{47}{2}} \\
&+ 4\,\frac{2\binom{4}{1}\binom{3}{1}\binom{40}{0} + \binom{4}{1}^2\binom{39}{0} - 3\binom{2}{2}\binom{45}{0}}{\binom{47}{2}} \\
&+ 3\,\frac{3\binom{3}{2}\binom{44}{0}}{\binom{47}{2}} + 2\,\frac{3\binom{3}{1}^2\binom{41}{0}}{\binom{47}{2}} + 1\,\frac{\binom{3}{1}\binom{6}{0}\binom{38}{1} + 2\binom{3}{2}\binom{44}{0} + \binom{4}{2}\binom{43}{0}}{\binom{47}{2}} \\
= \frac{703}{1{,}081} &\approx 0.650324, \qquad (17.4)
\end{aligned}$$

where we have omitted the term corresponding to no payout because it does not contribute.

Finally, if the player holds the three-card two-gap royal flush A♣-J♣-T♣,

$$\begin{aligned}
\mathrm{E}[R] = 800\,&\frac{\binom{2}{2}\binom{45}{0}}{\binom{47}{2}} + 6\,\frac{\left[\binom{9}{2} - \binom{2}{2}\binom{7}{0}\right]\binom{38}{0}}{\binom{47}{2}} + 4\,\frac{\binom{4}{1}\binom{3}{1}\binom{40}{0} - \binom{2}{2}\binom{45}{0}}{\binom{47}{2}} \\
&+ 3\,\frac{3\binom{3}{2}\binom{44}{0}}{\binom{47}{2}} + 2\,\frac{3\binom{3}{1}^2\binom{41}{0}}{\binom{47}{2}} + 1\,\frac{2\binom{3}{1}\binom{6}{0}\binom{38}{1} + \binom{4}{2}\binom{43}{0} + \binom{3}{2}\binom{44}{0}}{\binom{47}{2}} \\
= \frac{1{,}372}{1{,}081} &\approx 1.269195. \qquad (17.5)
\end{aligned}$$

While there do not seem to be any other reasonable strategies, one should not rely too heavily on one's intuition in these situations, so we have computed the conditional expectations in the other 27 cases, listing all 32 results in Table 17.2.

The five suggested strategies are indeed the five best. In particular, we conclude that holding A♣-J♣-T♣-9♣ is the optimal strategy, because it maximizes the player's conditional expected payout. This suggests that drawing to a four-card flush is marginally better than drawing to a three-card two-gap royal flush. However, one has to be careful when making such generalizations.

Let us consider a closely related example to illustrate this point. Suppose that the player is dealt A♣-J♣-T♣-9♣-6♢ (in any order). (The Q♢ in the preceding example has been replaced by the 6♢.) The analogues of the two best strategies in the preceding example are the following.

If the player holds the four-card flush A♣-J♣-T♣-9♣, his payout R from a one-unit bet has conditional expected value (17.3).

If the player holds the three-card two-gap royal flush A♣-J♣-T♣, (17.5) is replaced by

$$\mathrm{E}[R] = 800\,\frac{\binom{2}{2}\binom{45}{0}}{\binom{47}{2}} + 6\,\frac{\left[\binom{9}{2} - \binom{2}{2}\binom{7}{0}\right]\binom{38}{0}}{\binom{47}{2}} + 4\,\frac{\binom{4}{1}^2\binom{39}{0} - \binom{2}{2}\binom{45}{0}}{\binom{47}{2}}$$

Table 17.2 The 32 possible values of the player's conditional expected payout at Jacks or Better when he is dealt A♣-Q♢-J♣-T♣-9♣.

cards held					$k :=$ no. of cards drawn	conditional expected payout E	$\binom{47}{k}E$
A♣	Q♢	J♣	T♣	9♣	0	.000 000	0
A♣	Q♢	J♣	T♣		1	.531 915	25
A♣	Q♢	J♣		9♣	1	.191 489	9
A♣	Q♢		T♣	9♣	1	.127 660	6
A♣		J♣	T♣	9♣	1	1.276 596	60
	Q♢	J♣	T♣	9♣	1	.808 511	38
A♣	Q♢	J♣			2	.441 258	477
A♣	Q♢		T♣		2	.338 575	366
A♣	Q♢			9♣	2	.294 172	318
A♣		J♣	T♣		2	1.269 195	1,372
A♣		J♣		9♣	2	.493 987	534
A♣			T♣	9♣	2	.391 304	423
	Q♢	J♣	T♣		2	.427 382	462
	Q♢	J♣		9♣	2	.382 979	414
	Q♢		T♣	9♣	2	.280 296	303
		J♣	T♣	9♣	2	.650 324	703
A♣	Q♢				3	.464 693	7,535
A♣		J♣			3	.495 776	8,039
A♣			T♣		3	.361 640	5,864
A♣				9♣	3	.352 760	5,720
	Q♢	J♣			3	.482 455	7,823
	Q♢		T♣		3	.348 319	5,648
	Q♢			9♣	3	.339 439	5,504
		J♣	T♣		3	.391 243	6,344
		J♣		9♣	3	.382 362	6,200
			T♣	9♣	3	.266 482	4,321
A♣					4	.434 614	77,520
	Q♢				4	.450 784	80,404
		J♣			4	.436 722	77,896
			T♣		4	.272 279	48,565
				9♣	4	.275 822	49,197
(none)					5	.308 626	473,414

$$+ 3\,\frac{3\binom{3}{2}\binom{44}{0}}{\binom{47}{2}} + 2\,\frac{3\binom{3}{1}\binom{3}{1}\binom{41}{0}}{\binom{47}{2}} + 1\,\frac{2\binom{3}{1}\binom{6}{0}\binom{38}{1} + 2\binom{4}{2}\binom{43}{0}}{\binom{47}{2}}$$
$$= \frac{1{,}391}{1{,}081} \approx 1.286772. \tag{17.6}$$

None of the other 30 strategies is as good as these two. We conclude that the optimal strategy is to hold A♣-J♣-T♣.

Notice that the two best strategies (hold the four-card flush; hold the three-card two-gap royal flush) are ordered differently in the two examples. The reason is easy to see. The player's expected payout when holding the four-card flush is the same in the two examples. However, his expected payout when holding the three-card two-gap royal flush is smaller when he discards the Q♢ (first example) than when he discards the 6♢ (second example). Clearly, the absence of the Q♢ from the residual 47-card deck reduces the chance that the player will make a straight. (It also reduces the chance that he will make a pair of jacks or better.) Thus, the Q♢ is called a straight-penalty card.

More generally, when a particular discard reduces the probability that the player will make a straight or a flush (over what it would have been otherwise), that discard is called a *straight*- or a *flush-penalty card*. Another example may help to clarify this concept.

Suppose that the player is dealt A♣-Q♢-J♣-T♣-9♡ (in any order). (The 9♣ in the first example has been replaced by the 9♡.) The player can no longer hold a four-card flush, but the strategy of holding the three-card two-gap royal flush A♣-J♣-T♣ is still viable. In this case the player's payout R from a one-unit bet has conditional expected value

$$\mathrm{E}[R] = 800\,\frac{\binom{2}{2}\binom{45}{0}}{\binom{47}{2}} + 6\,\frac{\left[\binom{10}{2} - \binom{2}{2}\binom{8}{0}\right]\binom{37}{0}}{\binom{47}{2}} + 4\,\frac{\binom{3}{1}\binom{4}{1}\binom{40}{0} - \binom{2}{2}\binom{45}{0}}{\binom{47}{2}}$$
$$+ 3\,\frac{3\binom{3}{2}\binom{44}{0}}{\binom{47}{2}} + 2\,\frac{3\binom{3}{1}^2\binom{41}{0}}{\binom{47}{2}} + 1\,\frac{2\binom{3}{1}\binom{6}{0}\binom{38}{1} + \binom{4}{2}\binom{43}{0} + \binom{3}{2}\binom{44}{0}}{\binom{47}{2}}$$
$$= \frac{1{,}426}{1{,}081} \approx 1.319149. \tag{17.7}$$

Here the player's expected payout when holding the three-card two-gap royal flush is smaller when he discards the 9♣ (first example) than when he discards the 9♡ (third example). Thus, the 9♣ in the first example is a flush-penalty card.

It is now clear what must be done to determine the optimal strategy at Jacks or Better. We simply create a table analogous to Table 17.2 for each of the player's $\binom{52}{5} = 2{,}598{,}960$ possible initial hands, determine for each such hand which of the $2^5 = 32$ ways to play it maximizes his conditional expected payout, summarize the resulting player strategy, and average the conditional expectations thus obtained to evaluate the overall expected payout. Of course the variance of the payout and more generally its distribution would also be of interest.

Actually, we can reduce the amount of work required by nearly a factor of 20 by taking equivalence of initial hands into account, just as we did in Section 16.2. Let us call two initial hands *equivalent* if they have the same five denominations and if the corresponding denominations have the same suits after a permutation of (♣,♢,♡,♠). This is an equivalence relation in the sense of Theorem A.1.1. Thus, for example, the equivalence class containing A♣-A♢-A♡-K♣-Q♢ has $\binom{4}{3}\binom{3}{1}\binom{2}{1} = 24$ hands, the one containing A♣-A♢-A♡-K♣-Q♠ has $\binom{4}{3}\binom{3}{1} = 12$ hands, and the one containing A♣-A♢-A♡-K♠-Q♠ has $\binom{4}{3} = 4$ hands.

How many equivalence classes are there of a given size associated with a particular set of denominations? This question can be answered on a case-by-case basis, so we consider just one such case. Consider a hand with five distinct denominations m_1, m_2, m_3, m_4, m_5 with $14 \geq m_1 > m_2 > m_3 > m_4 > m_5 \geq 2$. (Here 14, 13, 12, 11 correspond to A, K, Q, J. There are $\binom{13}{5} = 1{,}287$ ways to choose the denominations.) We number the suits of denominations m_1, m_2, m_3, m_4, m_5 by $n_1, n_2, n_3, n_4, n_5 \in \{1, 2, 3, 4\}$. Since we are concerned only with equivalence classes, we choose n_1, n_2, n_3, n_4, n_5 successively, using the smallest available integer for each suit that does not appear in a higher denomination. Thus,

$$\begin{aligned} n_1 &= 1 \\ n_2 &\leq n_1 + 1 \\ n_3 &\leq \max(n_1, n_2) + 1 \\ n_4 &\leq \max(n_1, n_2, n_3) + 1 \\ n_5 &\leq \max(n_1, n_2, n_3, n_4) + 1. \end{aligned} \tag{17.8}$$

It is easy to see that there is a one-to-one correspondence between the set of such $(n_1, n_2, n_3, n_4, n_5)$ and the set of equivalence classes of hands with denominations $(m_1, m_2, m_3, m_4, m_5)$. By direct enumeration (rather than by combinatorial analysis) we find that there are 51 equivalence classes. See Table 17.3, which also includes the other types of hands.

Table 17.3 shows that there are exactly

$$\binom{13}{5}51 + \binom{13}{1,3,9}20 + \binom{13}{2,1,10}8 + \binom{13}{1,2,10}5 + \binom{13}{1,1,11}3 = 134{,}459 \tag{17.9}$$

equivalence classes. As a check, we compute the total number of hands by summing the sizes of the equivalence classes:

$$\begin{aligned} &1{,}287(1 \cdot 4 + 15 \cdot 12 + 35 \cdot 24) + 2{,}860(8 \cdot 12 + 12 \cdot 24) \\ &\quad + 858(1 \cdot 4 + 7 \cdot 12 + 5 \cdot 24) + 156(1 \cdot 4 + 2 \cdot 12) = 2{,}598{,}960. \end{aligned} \tag{17.10}$$

Table 17.3 List of equivalence classes of initial player hands in Jacks or Better, together with the size of each equivalence class. The hand A♣-Q♢-J♣-T♣-9♣, for example, belongs to the equivalence class A-Q-J-T-9 $(1,2,1,1,1)$, as do 11 other hands with the same denominations.

five distinct denominations (a,b,c,d,e): $\binom{13}{5} = 1{,}287$ ways
(includes hands ranked no pair, straight, flush, straight flush, royal flush)

$(1,1,1,1,1)$	4	$(1,1,2,3,3)$	24	$(1,2,2,1,2)$	12	$(1,2,3,2,1)$	24
$(1,1,1,1,2)$	12	$(1,1,2,3,4)$	24	$(1,2,2,1,3)$	24	$(1,2,3,2,2)$	24
$(1,1,1,2,1)$	12	$(1,2,1,1,1)$	12	$(1,2,2,2,1)$	12	$(1,2,3,2,3)$	24
$(1,1,1,2,2)$	12	$(1,2,1,1,2)$	12	$(1,2,2,2,2)$	12	$(1,2,3,2,4)$	24
$(1,1,1,2,3)$	24	$(1,2,1,1,3)$	24	$(1,2,2,2,3)$	24	$(1,2,3,3,1)$	24
$(1,1,2,1,1)$	12	$(1,2,1,2,1)$	12	$(1,2,2,3,1)$	24	$(1,2,3,3,2)$	24
$(1,1,2,1,2)$	12	$(1,2,1,2,2)$	12	$(1,2,2,3,2)$	24	$(1,2,3,3,3)$	24
$(1,1,2,1,3)$	24	$(1,2,1,2,3)$	24	$(1,2,2,3,3)$	24	$(1,2,3,3,4)$	24
$(1,1,2,2,1)$	12	$(1,2,1,3,1)$	24	$(1,2,2,3,4)$	24	$(1,2,3,4,1)$	24
$(1,1,2,2,2)$	12	$(1,2,1,3,2)$	24	$(1,2,3,1,1)$	24	$(1,2,3,4,2)$	24
$(1,1,2,2,3)$	24	$(1,2,1,3,3)$	24	$(1,2,3,1,2)$	24	$(1,2,3,4,3)$	24
$(1,1,2,3,1)$	24	$(1,2,1,3,4)$	24	$(1,2,3,1,3)$	24	$(1,2,3,4,4)$	24
$(1,1,2,3,2)$	24	$(1,2,2,1,1)$	12	$(1,2,3,1,4)$	24		

one pair (a,a,b,c,d): $\binom{13}{1,3,9} = 2{,}860$ ways

$(1,2,1,1,1)$	12	$(1,2,1,2,3)$	24	$(1,2,3,1,1)$	24	$(1,2,3,3,3)$	12
$(1,2,1,1,2)$	12	$(1,2,1,3,1)$	24	$(1,2,3,1,2)$	24	$(1,2,3,3,4)$	12
$(1,2,1,1,3)$	24	$(1,2,1,3,2)$	24	$(1,2,3,1,3)$	24	$(1,2,3,4,1)$	24
$(1,2,1,2,1)$	12	$(1,2,1,3,3)$	24	$(1,2,3,1,4)$	24	$(1,2,3,4,3)$	12
$(1,2,1,2,2)$	12	$(1,2,1,3,4)$	24	$(1,2,3,3,1)$	24	$(1,2,3,4,4)$	12

two pair (a,a,b,b,c): $\binom{13}{2,1,10} = 858$ ways

$(1,2,1,2,1)$	12	$(1,2,1,3,1)$	24	$(1,2,1,3,3)$	24	$(1,2,3,4,1)$	12
$(1,2,1,2,3)$	12	$(1,2,1,3,2)$	24	$(1,2,1,3,4)$	24	$(1,2,3,4,3)$	12

three of a kind (a,a,a,b,c): $\binom{13}{1,2,10} = 858$ ways

$(1,2,3,1,1)$	12	$(1,2,3,1,4)$	12	$(1,2,3,4,4)$	4
$(1,2,3,1,2)$	24	$(1,2,3,4,1)$	12		

full house (a,a,a,b,b): $\binom{13}{1,1,11} = 156$ ways

$(1,2,3,1,2)$	12	$(1,2,3,1,4)$	12

four of a kind (a,a,a,a,b): $\binom{13}{1,1,11} = 156$ ways

$(1,2,3,4,1)$	4

Needless to say, a computer is a necessity for this kind of problem. Our program methodically cycles through each of the 134,459 equivalence classes. For each one it computes the 32 conditional expectations and determines which is largest and if it is uniquely the largest. It stores this information in a file as it proceeds. Finally, it computes the payout distribution under the optimal strategy, first for each equivalence class and then for the game as a whole.

Let us now consider the issue of uniqueness. In one obvious case the optimal strategy is nonunique: If a player is dealt four of a kind, he may hold or discard the card of the odd denomination with no effect. This accounts for 156 equivalence classes or 624 hands. There is only one other situation for which uniqueness fails. With K-Q-J-T-T it is optimal to discard one of the tens—it does not matter which one—unless three or more of the cards are of the same suit. Of the 20 equivalence classes for this set of denominations, 3 of size 12 and 9 of size 24 have nonunique optimal strategies. This accounts for another 12 equivalence classes or 252 hands. However, it is important to note that the payout distribution is unaffected by the choice of optimal strategy in each of these cases of nonuniqueness.

Thus, the payout distribution for Jacks or Better played optimally is uniquely determined, and we display it in Table 17.4. Here it is worth giving exact results. A common denominator (not necessarily the least one) is

$$\text{l.c.m.}\left\{\binom{52}{5}\binom{47}{k} : k = 0, 1, 2, 3, 4, 5\right\} = \binom{52}{5}\binom{47}{5}5, \tag{17.11}$$

where l.c.m. stands for least common multiple. In fact, the least common denominator is this number divided by 12.

It follows that the mean payout under optimal play is

$$\frac{1{,}653{,}526{,}326{,}983}{1{,}661{,}102{,}543{,}100} \approx 0.995439043695, \tag{17.12}$$

while the variance of the payout is 19.514676427086. Thus, the house has a slight advantage (less than half of one percent) over the optimal Jacks or Better player.

There remains an important issue that has not yet been addressed. What exactly is the optimal strategy at Jacks or Better? Our computer program provides one possible answer: specific instructions for each of the 134,459 equivalence classes. However, what we need is something simpler, a strategy that can actually be memorized. The usual approach is to construct a so-called *hand-rank table*. Each of the various types of holdings is ranked according to its conditional expectation (which varies slightly with the cards discarded). One then simply finds the highest-ranked hand in the table that is applicable to the hand in question. Only relatively recently has a hand-rank table been found for Jacks or Better that reproduces the optimal strategy precisely, and that table is presented as Table 17.5.

Table 17.4 The distribution of the payout R from a one-unit bet on Jacks or Better. Assumes maximum-coin bet and optimal drawing strategy.

result	R	probability	probability $\times \binom{52}{5}\binom{47}{5}5/12$
royal flush	800	.000 024 758 268	41,126,022
straight flush	50	.000 109 309 090	181,573,608
four of a kind	25	.002 362 545 686	3,924,430,647
full house	9	.011 512 207 336	19,122,956,883
flush	6	.011 014 510 968	18,296,232,180
straight	4	.011 229 367 241	18,653,130,482
three of a kind	3	.074 448 698 571	123,666,922,527
two pair	2	.129 278 902 480	214,745,513,679
high pair (jacks or better)	1	.214 585 031 126	356,447,740,914
other	0	.545 434 669 233	906,022,916,158
sum		1.000 000 000 000	1,661,102,543,100

Let us consider the example, A♣-Q♢-J♣-T♣-9♣, discussed near the beginning of this section. Recall that we identified the five most promising strategies: (a) hold the four-card one-gap straight, (b) hold the four-card open-ended straight, (c) hold the four-card flush, (d) hold the three-card open-ended straight flush, and (e) hold the three-card two-gap royal flush.

(a) is not in the table, (b) is ranked 12th, (c) is ranked seventh, (d) is ranked 13th ($s = 3$ and $h = 1$), and (e) is ranked eighth. Holding the four-card flush ranks highest and is therefore the correct strategy, as we have already seen.

Finally, we reconsider the closely related example A♣-J♣-T♣-9♣-6♢. The two best strategies are (c) and (e) from the preceding example.

(c) is now ranked ninth, while (e) is still ranked eighth. Holding the three-card two-gap royal flush ranks highest and is therefore the correct strategy, as we have already seen. This emphasizes the fact the our hand-rank table is sensitive enough to take penalty cards into account.

17.2 Deuces Wild

As with Jacks or Better, the *Deuces Wild* video-poker player receives five cards face up on the screen, with each of the $\binom{52}{5} = 2{,}598{,}960$ possible hands equally likely. For each card, he must then decide whether to hold or discard that card. Thus, there are $2^5 = 32$ ways to play the hand. If he discards

Table 17.5 The optimal strategy at Jacks or Better. Choose the applicable strategy ranked highest in the table, holding only the cards listed in that strategy. If none applies, draw five new cards. Abbreviations: RF = royal flush, SF = straight flush, 4K = four of a kind, FH = full house, F = flush, S = straight, 3K = three of a kind, 2P = two pair, HP = high pair (jacks or better), LP = low pair. n-RF, n-SF, n-F, and n-S refer to n-card hands that have the potential to become RF, SF, F, and S, respectively. 3-3K, 4-2P, 2-HP, and 2-LP have a slightly different meaning: For example, 3-3K is a 3-card three of a kind, i.e., the potential is already realized. A, K, Q, J, T denote ace, king, queen, jack, and ten. H denotes any high card (A, K, Q, J). If two or more cards are italicized, that indicates that they are of the same suit. s is the number of straights, disregarding suits, that can be made from the hand, and h denotes the number of high cards in the hand. fp, sp, and 9sp denote flush penalty, straight penalty, and 9 straight penalty.

rank	description	rank	description
1	5-RF, 5-SF, 5-4K, or 5-FH	17	2-RF: *AH* or *KH*
2	3-3K	18	3-SF: $s+h=2$, no sp
3	4-2P or 4-RF	19	4-S: AHHT or KQJ9
4	5-F or 5-S	20	3-SF: $s+h=2$
5	4-SF	21	3-S: KQJ
6	2-HP	22	2-S: QJ
7	4-F: *AHTx* + K, Q, J, or T	23	2-S: KJ if *JT* fp
8	3-RF	24	2-RF: *JT*
9	4-F	25	2-S: KH
10	4-S: KQJT	26	2-S: AQ if *QT* fp
11	2-LP	27	2-RF: *QT*
12	4-S: 5432–QJT9	28	2-S: AH
13	3-SF: $s+h\geq 3$	29	1-RF: K if *KT* fp and 9sp
14	4-S: AK*QJ* if *QJ* fp or 9p	30	2-RF: *KT*
15	2-RF: *QJ*	31	1-RF: A, K, Q, or J
16	4-S: AKQJ	32	3-SF: $s+h=1$

k cards, he is dealt k new cards, with each of the $\binom{47}{k}$ possibilities equally likely. The player then receives his payout, which depends on the amount he bet and the rank of his final hand.

In Deuces Wild, as the name suggests, the four deuces (i.e., twos) are wild cards, and this affects the payout schedule. A *wild card* is a card that can play the role of any one of the 52 cards, even one that already appears in the hand. For example, A♣-A♢-A♡-A♠-2♣ counts as five of a kind. When wild cards are present in a hand, there may be more than one way to interpret the hand. The interpretation with the highest payout is the one that applies.

For example, A♣-2♣-2♢-2♡-2♠ counts as four deuces, not five of a kind, because four deuces pays more than five of a kind. Similarly, A♣-A♢-K♣-2♣-2♢ counts as four of a kind, not a full house.

The payout schedule for full-pay Deuces Wild is shown in Table 17.6. A *wild royal flush* is a royal flush with at least one wild card. A *natural royal flush* is a royal flush with no wild cards. We again use the term "full-pay" to emphasize the fact that there are similar machines with less favorable payout schedules. It should also be mentioned that, typically, to qualify for the 800 for 1 payout on a natural royal flush, the player must bet five coins. We assume that the player does this, and we define the total value of these five coins to be one unit.

Table 17.6 The full-pay Deuces Wild payoff odds and pre-draw frequencies.

rank	payoff odds	number of ways
natural royal flush	800 for 1	4
four deuces	200 for 1	48
wild royal flush	25 for 1	480
five of a kind	15 for 1	624
straight flush	9 for 1	2,068
four of a kind	5 for 1	31,552
full house	3 for 1	12,672
flush	2 for 1	14,472
straight	2 for 1	62,232
three of a kind	1 for 1	355,080
other	0 for 1	2,119,728
total		2,598,960

In Tables 17.7 and 17.8 we show how the pre-draw frequencies in Table 17.6 were evaluated. The key is to first classify each poker hand according to the number of deuces it contains (4, 3, 2, 1, or 0). Then it is relatively straightforward to count the numbers of hands of each type within each of these five classes. For example, to describe a straight with two deuces, we first specify the two deuces $\left[\binom{4}{2}\text{ ways}\right]$. Then we specify the lowest non-deuce denomination in the straight. If it is an ace, then one of the deuces must be used as a deuce and the hand must contain two of the three denominations 3, 4, and 5 $\left[\binom{1}{1}\binom{3}{2}\text{ ways}\right]$. If it is 3–T, then the hand must contain two of the next four denominations $\left[\binom{8}{1}\binom{4}{2}\text{ ways}\right]$. If it is a jack, the hand must contain two of the denomination Q, K, A $\left[\binom{1}{1}\binom{3}{2}\text{ ways}\right]$. If it is a queen, the hand must contain both of the denominations K and A $\left[\binom{1}{1}\binom{2}{2}\text{ ways}\right]$. Finally, we must specify the suits of the three non-deuces $\left[\binom{4}{1}^3 - \binom{4}{1}\text{ ways}\right]$, which cannot be the same or we would have a straight flush or a wild royal flush.

The primary issue in Deuces Wild is to determine, given the player's initial five-card hand, which cards should be held. Let us consider an example. Suppose that the player is dealt K♣-Q♢-J♢-T♢-8♢ (in any order, of course). There are three plausible strategies.

If the player holds the four-card open-ended straight K♣-Q♢-J♢-T♢, his payout R from a one-unit bet has conditional expected value

$$\mathrm{E}[R] = 2\left(\frac{12}{47}\right) + 0\left(\frac{35}{47}\right) = \frac{24}{47} \approx 0.510638. \tag{17.13}$$

For convenience we do not write out the conditioning event, namely "player is dealt K♣-Q♢-J♢-T♢-8♢ and holds K♣-Q♢-J♢-T♢."

If the player holds the four-card one-gap straight flush Q♢-J♢-T♢-8♢, his payout R from a one-unit bet has conditional expected value

$$\mathrm{E}[R] = 9\left(\frac{5}{47}\right) + 2\left(\frac{7}{47}\right) + 2\left(\frac{3}{47}\right) + 0\left(\frac{32}{47}\right) = \frac{65}{47} \approx 1.382979. \tag{17.14}$$

If the player holds the three-card royal flush Q♢-J♢-T♢, his payout R on a one-unit bet has conditional expected value

$$\begin{aligned}\mathrm{E}[R] = {} & 800\,\frac{\binom{2}{2}\binom{45}{0}}{\binom{47}{2}} + 25\,\frac{\binom{4}{2}\binom{43}{0} + \binom{4}{1}\binom{2}{1}\binom{41}{0}}{\binom{47}{2}} + 9\,\frac{\binom{4}{1}\binom{1}{1}\binom{42}{0} + \binom{4}{0}\binom{2}{2}\binom{41}{0}}{\binom{47}{2}} \\ & + 2\,\frac{\binom{4}{1}\binom{5}{1}\binom{38}{0} + \binom{4}{0}\left[\binom{8}{2} - 2\binom{2}{2}\binom{6}{0}\right]\binom{35}{0}}{\binom{47}{2}} \\ & + 2\,\frac{\binom{4}{1}\binom{11}{1}\binom{32}{0} + \binom{4}{0}\left[3\binom{4}{1}\binom{3}{1}\binom{36}{0} - 2\binom{2}{2}\binom{41}{0}\right]}{\binom{47}{2}} \\ & + 1\,\frac{\binom{4}{1}\binom{9}{1}\binom{34}{0} + \binom{4}{0}3\binom{3}{2}\binom{40}{0}}{\binom{47}{2}} \\ = {} & \frac{1{,}488}{1{,}081} \approx 1.376503. \end{aligned} \tag{17.15}$$

While there do not seem to be any other reasonable strategies, one should not rely too heavily on one's intuition in these situations, so we have computed the conditional expectations in the other 29 cases, listing all 32 results in Table 17.9.

The three suggested strategies are indeed the three best. In particular, we conclude that holding Q♢-J♢-T♢-8♢ is the optimal strategy, because it maximizes the player's conditional expected payout. This suggests that drawing to a four-card one-gap straight flush is marginally better than drawing to a three-card royal flush. However, one has to be careful when making such generalizations.

Table 17.7 The full-pay Deuces Wild pre-draw frequencies.

deuces	rank	number of ways	
4	four deuces	$\binom{4}{4}\binom{48}{1}$	48
3	wild royal flush	$\binom{4}{3}\binom{5}{2}\binom{4}{1}$	160
3	five of a kind	$\binom{4}{3}\binom{12}{1}\binom{4}{2}$	288
3	straight flush	$\binom{4}{3}\left[\binom{1}{1}\binom{3}{1}+\binom{7}{1}\binom{4}{1}\right]\binom{4}{1}$	496
3	four of a kind	$\binom{4}{3}\binom{48}{2}$ − subtotal	3,568
2	wild royal flush	$\binom{4}{2}\binom{5}{3}\binom{4}{1}$	240
2	five of a kind	$\binom{4}{2}\binom{12}{1}\binom{4}{3}$	288
2	straight flush	$\binom{4}{2}\left[\binom{1}{1}\binom{3}{2}+\binom{7}{1}\binom{4}{2}\right]\binom{4}{1}$	1,080
2	four of a kind	$\binom{4}{2}\binom{12}{1}\binom{11}{1}\binom{4}{2}\binom{4}{1}$	19,008
2	flush	$\binom{4}{2}\left[\binom{12}{3}-\binom{5}{3}-\binom{1}{1}\binom{3}{2}-\binom{7}{1}\binom{4}{2}\right]\binom{4}{1}$	3,960
2	straight	$\binom{4}{2}\left[\binom{1}{1}\binom{3}{2}+\binom{8}{1}\binom{4}{2}+\binom{1}{1}\binom{3}{2}\right.$	
		$\left.+\binom{1}{1}\binom{2}{2}\right]\left[\binom{4}{1}^3-\binom{4}{1}\right]$	19,800
2	three of a kind	$\binom{4}{2}\binom{48}{3}$ − subtotal	59,400
1	wild royal flush	$\binom{4}{1}\binom{5}{4}\binom{4}{1}$	80
1	five of a kind	$\binom{4}{1}\binom{12}{1}\binom{4}{4}$	48
1	straight flush	$\binom{4}{1}\left[\binom{1}{1}\binom{4}{4}+\binom{7}{1}\binom{4}{3}\right]\binom{4}{1}$	464
1	four of a kind	$\binom{4}{1}\binom{12}{1}\binom{11}{1}\binom{4}{3}\binom{4}{1}$	8,448
1	full house	$\binom{4}{1}\binom{12}{2}\binom{4}{2}^2$	9,504
1	flush	$\binom{4}{1}\left[\binom{12}{4}-\binom{5}{4}-\binom{1}{1}\binom{4}{4}-\binom{7}{1}\binom{4}{3}\right]\binom{4}{1}$	7,376
1	straight	$\binom{4}{1}\left[\binom{1}{1}\binom{3}{3}+\binom{8}{1}\binom{4}{3}\right.$	
		$\left.+\binom{1}{1}\binom{3}{3}\right]\left[\binom{4}{1}^4-\binom{4}{1}\right]$	34,272
1	three of a kind	$\binom{4}{1}\binom{12}{1}\binom{11}{2}\binom{4}{2}\binom{4}{1}^2$	253,440
1	one pair*	$\binom{4}{1}\binom{48}{4}$ − subtotal	464,688

continued in Table 17.8

*no payout

Let us consider a closely related example to illustrate this point. Suppose that the player is dealt Q♢-J♢-T♢-8♢-7♣ (in any order). (The K♣ in the preceding example has been replaced by the 7♣.) The analogues of the two best strategies in the preceding example are the following.

If the player holds the four-card one-gap straight flush Q♢-J♢-T♢-8♢, his payout R from a one-unit bet has conditional expected value (17.14).

Table 17.8 Continuation of Table 17.7: The full-pay Deuces Wild pre-draw frequencies.

deuces	rank	number of ways	
0	natural royal flush	$\binom{4}{0}\binom{5}{5}\binom{4}{1}$	4
0	straight flush	$\binom{4}{0}\binom{7}{1}\binom{4}{4}\binom{4}{1}$	28
0	four of a kind	$\binom{4}{0}\binom{12}{1}\binom{11}{1}\binom{4}{4}\binom{4}{1}$	528
0	full house	$\binom{4}{0}\binom{12}{1}\binom{11}{1}\binom{4}{3}\binom{4}{2}$	3,168
0	flush	$\binom{4}{0}\left[\binom{12}{5}-\binom{5}{5}-\binom{7}{1}\binom{4}{4}\right]\binom{4}{1}$	3,136
0	straight	$\binom{4}{0}\binom{8}{1}\left[\binom{4}{1}^5-\binom{4}{1}\right]$	8,160
0	three of a kind	$\binom{4}{0}\binom{12}{1}\binom{11}{2}\binom{4}{3}\binom{4}{1}^2$	42,240
0	no payout	$\binom{4}{0}\binom{48}{5}$ − subtotal	1,655,040
total (Tables 17.7 and 17.8)			2,598,960

If the player holds the three-card royal flush Q♢-J♢-T♢, his payout R from a one-unit bet has conditional expected value

$$\begin{aligned}\mathrm{E}[R] &= 800\,\frac{\binom{2}{2}\binom{45}{0}}{\binom{47}{2}} + 25\,\frac{\binom{4}{2}\binom{43}{0}+\binom{4}{1}\binom{2}{1}\binom{41}{0}}{\binom{47}{2}} + 9\,\frac{\binom{4}{1}\binom{1}{1}\binom{42}{0}+\binom{4}{0}\binom{2}{2}\binom{41}{0}}{\binom{47}{2}}\\
&\quad + 2\,\frac{\binom{4}{1}\binom{5}{1}\binom{38}{0}+\binom{4}{0}\left[\binom{8}{2}-2\binom{2}{2}\binom{6}{0}\right]\binom{35}{0}}{\binom{47}{2}}\\
&\quad + 2\,\frac{\binom{4}{1}\binom{12}{1}\binom{31}{0}+\binom{4}{0}\left[2\binom{4}{1}^2\binom{35}{0}+\binom{4}{1}\binom{3}{1}\binom{36}{0}-2\binom{2}{2}\binom{41}{0}\right]}{\binom{47}{2}}\\
&\quad + 1\,\frac{\binom{4}{1}\binom{9}{1}\binom{34}{0}+\binom{4}{0}3\binom{3}{2}\binom{40}{0}}{\binom{47}{2}}\\
&= \frac{1{,}512}{1{,}081} \approx 1.398705. \end{aligned} \tag{17.16}$$

Table 17.9 The 32 possible values of the player's conditional expected payout at Deuces Wild when he is dealt K♣-Q♢-J♢-T♢-8♢.

cards held					$k :=$ no. of cards drawn	conditional expected payout E	$\binom{47}{k}E$
K♣	Q♢	J♢	T♢	8♢	0	.000 000	0
K♣	Q♢	J♢	T♢		1	.510 638	24
K♣	Q♢	J♢		8♢	1	.000 000	0
K♣	Q♢		T♢	8♢	1	.000 000	0
K♣		J♢	T♢	8♢	1	.000 000	0
	Q♢	J♢	T♢	8♢	1	1.382 979	65
K♣	Q♢	J♢			2	.178 538	193
K♣	Q♢		T♢		2	.178 538	193
K♣	Q♢			8♢	2	.047 179	51
K♣		J♢	T♢		2	.178 538	193
K♣		J♢		8♢	2	.047 179	51
K♣			T♢	8♢	2	.047 179	51
	Q♢	J♢	T♢		2	1.376 503	1,488
	Q♢	J♢		8♢	2	.295 097	319
	Q♢		T♢	8♢	2	.295 097	319
		J♢	T♢	8♢	2	.377 428	408
K♣	Q♢				3	.169 164	2,743
K♣		J♢			3	.169 164	2,743
K♣			T♢		3	.169 164	2,743
K♣				8♢	3	.126 981	2,059
	Q♢	J♢			3	.238 606	3,869
	Q♢		T♢		3	.238 606	3,869
	Q♢			8♢	3	.178 292	2,891
		J♢	T♢		3	.263 275	4,269
		J♢		8♢	3	.202 960	3,291
			T♢	8♢	3	.233 117	3,780
K♣					4	.250 397	44,662
	Q♢				4	.236 633	42,207
		J♢			4	.245 805	43,843
			T♢		4	.257 220	45,879
				8♢	4	.323 817	48,117
(none)					5	.323 817	496,716

None of the other 30 strategies is as good as these two. We conclude that the optimal strategy is to hold Q♢-J♢-T♢.

Notice that the two best strategies (hold the four-card one-gap straight flush; hold the three-card royal flush) are ordered differently in the two examples. The reason is easy to see. The player's expected payout when holding the four-card one-gap straight flush is the same in the two examples. However, his expected payout when holding the three-card royal flush is smaller when he discards the K♣ (first example) than when he discards the 7♣ (second example). Clearly, the absence of the K♣ from the residual 47-card deck reduces the chance that the player will make a straight. Thus, the K♣ is called a straight-penalty card.

More generally, when a particular discard reduces the probability that the player will make a straight or a flush (over what it would have been otherwise), that discard is called a *straight-* or a *flush-penalty card*. There are other types of penalty cards as well, as the following example indicates.

Suppose that the player is dealt five of a kind with three deuces. Should he hold the pat five of a kind or discard the nondeuce pair to draw for the fourth deuce? The only question is whether the latter strategy provides a conditional expected payout from a one-unit bet of more than 15 units, which is the guaranteed payout for the pat five of a kind. Surprisingly, the answer depends on the denomination of the nondeuce pair. For example, if the nondeuce pair has denomination 9, the player's payout R has conditional expected value

$$\begin{aligned} \mathrm{E}[R] &= 5 + 195\,\frac{\binom{1}{1}\binom{46}{1}}{\binom{47}{2}} + 20\,\frac{\binom{5}{2}\binom{4}{1}}{\binom{47}{2}} + 10\,\frac{11\binom{4}{2}\binom{43}{0} + \binom{2}{2}\binom{45}{0}}{\binom{47}{2}} \\ &\quad + 4\,\frac{2\binom{4}{1}\binom{4}{1} + 4\big[\binom{2}{1}\binom{4}{1} + \binom{2}{1}\binom{3}{1}\big] + \binom{2}{1}\binom{4}{1} + \binom{4}{1}\binom{3}{1}}{\binom{47}{2}} \\ &= \frac{16{,}277}{1{,}081} \approx 15.057354. \qquad (17.17) \end{aligned}$$

For a second example, if the nondeuce pair has denomination ten, the player's payout R has conditional expected value

$$\begin{aligned} \mathrm{E}[R] &= 5 + 195\,\frac{\binom{1}{1}\binom{46}{1}}{\binom{47}{2}} + 20\,\frac{\binom{1}{0}\binom{4}{2}\binom{4}{1} + \binom{1}{1}\binom{4}{1}\binom{2}{1}}{\binom{47}{2}} + 10\,\frac{11\binom{4}{2}\binom{43}{0} + \binom{2}{2}\binom{45}{0}}{\binom{47}{2}} \\ &\quad + 4\,\frac{3\binom{4}{1}\binom{4}{1} + 4\big[\binom{2}{1}\binom{4}{1} + \binom{2}{1}\binom{3}{1}\big] + \binom{4}{1}\binom{3}{1}}{\binom{47}{2}} \\ &= \frac{16{,}149}{1{,}081} \approx 14.938945. \qquad (17.18) \end{aligned}$$

In evaluating both conditional expectations, we noted that the player is assured a 5-unit payout, so we added to this guaranteed amount the additional contributions from hands better than four of a kind. (This allowed us to avoid counting the number of four-of-a-kind hands.)

Thus, we find that the optimal strategy with a three-deuce hand of five 9s is to hold only the deuces, while the optimal strategy with a three-deuce hand of five tens is to hold all five cards. In the case of the three-deuce hand of five tens, the two tens, if discarded, can be regarded as wild-royal-flush-penalty cards.

It is now clear what must be done to determine the optimal strategy at Deuces Wild. We simply create a table analogous to Table 17.9 for each of the player's $\binom{52}{5} = 2{,}598{,}960$ possible initial hands, determine for each such hand which of the $2^5 = 32$ ways to play it maximizes his conditional expected payout, summarize the resulting player strategy, and average the conditional expectations thus obtained to evaluate the overall expected payout.

Actually, we can reduce the amount of work required by more than a factor of 20 by taking equivalence of initial hands into account, much as we did in Section 17.1. Let us call two initial hands *equivalent* if they have the same five denominations and if the corresponding nondeuce denominations have the same suits after a permutation of (♣,♢,♡,♠). The reason for the word "nondeuce" in the definition is that the suits of deuces do not matter. This is an equivalence relation in the sense of Theorem A.1.1. Thus, for example, the equivalence class containing T♣-T♢-2♣-2♢-2♡ has $\binom{4}{2}\binom{4}{3} = 24$ hands, including T♣-T♢-2♢-2♡-2♠. (These two hands are not equivalent under the equivalence relation of Section 17.1.)

Table 17.10 shows that there are exactly

$$\begin{aligned}
&\binom{12}{5}51 + \binom{12}{1,3,8}20 + \binom{12}{2,1,9}8 + \binom{12}{1,2,9}5 + \binom{12}{1,1,10}3 \\
&\quad + \binom{12}{4}15 + \binom{12}{1,2,9}6 + \binom{12}{2}3 + \binom{12}{1,1,10}2 + \binom{12}{1}1 \\
&\quad + \binom{12}{3}5 + \binom{12}{1,1,10}2 + \binom{12}{1}1 + \binom{12}{2}2 + \binom{12}{1}2 = 102{,}359
\end{aligned} \tag{17.19}$$

equivalence classes. As a check, we compute the total number of hands by summing the sizes of the equivalence classes:

$$\begin{aligned}
&792(1\cdot 4 + 15\cdot 12 + 35\cdot 24) + 1{,}980(8\cdot 12 + 12\cdot 24) \\
&\quad + 660(1\cdot 4 + 7\cdot 12 + 5\cdot 24) + 132(1\cdot 4 + 2\cdot 12) \\
&\quad + 495(1\cdot 16 + 7\cdot 48 + 7\cdot 96) + 660(4\cdot 48 + 2\cdot 96) \\
&\quad + 66(2\cdot 24 + 1\cdot 96) + 132(1\cdot 16 + 1\cdot 48) + 12(1\cdot 4) \\
&\quad + 220(1\cdot 24 + 3\cdot 72 + 1\cdot 144) + 132(2\cdot 72) + 12(1\cdot 24) \\
&\quad + 66(1\cdot 16 + 1\cdot 48) + 12(1\cdot 4 + 1\cdot 24) = 2{,}598{,}960.
\end{aligned} \tag{17.20}$$

Again, a computer is a necessity for this kind of problem. Our program methodically cycles through each of the 102,359 equivalence classes. For each

Table 17.10 List of equivalence classes of initial player hands in Deuces Wild, together with the size of each equivalence class. a, b, c, d represent distinct nondeuce denominations.

no deuces: see Table 17.3[1] on p. 551							
$(2, a, b, c, d)$: $\binom{12}{4} = 495$ ways							
$(*, 1, 1, 1, 1)$	16	$(*, 1, 1, 2, 3)$	96	$(*, 1, 2, 2, 1)$	48	$(*, 1, 2, 3, 2)$	96
$(*, 1, 1, 1, 2)$	48	$(*, 1, 2, 1, 1)$	48	$(*, 1, 2, 2, 2)$	48	$(*, 1, 2, 3, 3)$	96
$(*, 1, 1, 2, 1)$	48	$(*, 1, 2, 1, 2)$	48	$(*, 1, 2, 2, 3)$	96	$(*, 1, 2, 3, 4)$	96
$(*, 1, 1, 2, 2)$	48	$(*, 1, 2, 1, 3)$	96	$(*, 1, 2, 3, 1)$	96		
$(2, a, a, b, c)$: $\binom{12}{1,2,10} = 660$ ways							
$(*, 1, 2, 1, 1)$	48	$(*, 1, 2, 1, 3)$	96	$(*, 1, 2, 3, 3)$	48		
$(*, 1, 2, 1, 2)$	48	$(*, 1, 2, 3, 1)$	96	$(*, 1, 2, 3, 4)$	48		
$(2, a, a, b, b)$: $\binom{12}{2} = 66$ ways							
$(*, 1, 2, 1, 2)$	24	$(*, 1, 2, 1, 3)$	96	$(*, 1, 2, 3, 4)$	24		
$(2, a, a, a, b)$: $\binom{12}{1,1,10} = 132$ ways							
$(*, 1, 2, 3, 1)$	48	$(*, 1, 2, 3, 4)$	16				
$(2, a, a, a, a)$: $\binom{12}{1} = 12$ ways							
$(*, 1, 2, 3, 4)$	4						
$(2, 2, a, b, c)$: $\binom{12}{3} = 220$ ways							
$(*, *, 1, 1, 1)$	24	$(*, *, 1, 2, 1)$	72	$(*, *, 1, 2, 3)$	144		
$(*, *, 1, 1, 2)$	72	$(*, *, 1, 2, 2)$	72				
$(2, 2, a, a, b)$: $\binom{12}{1,1,10} = 132$ ways							
$(*, *, 1, 2, 1)$	72	$(*, *, 1, 2, 3)$	72				
$(2, 2, a, a, a)$: $\binom{12}{1} = 12$ ways							
$(*, *, 1, 2, 3)$	24						
$(2, 2, 2, a, b)$: $\binom{12}{2} = 66$ ways							
$(*, *, *, 1, 1)$	16	$(*, *, *, 1, 2)$	48				
$(2, 2, 2, a, a)$: $\binom{12}{1} = 12$ ways							
$(*, *, *, 1, 2)$	24						
$(2, 2, 2, 2, a)$: $\binom{12}{1} = 12$ ways							
$(*, *, *, *, 1)$	4						

[1]except replace 13 by 12 in the multinomial coefficients, and adjust the corresponding partitions of 13

one it computes the 32 conditional expectations and determines which is largest and if it is uniquely the largest. It stores this information in a file as it proceeds. Finally, it computes the payout distribution under the optimal strategy, first for each equivalence class and then for the game as a whole.

Let us now consider the issue of uniqueness, which is more complicated than with Jacks or Better. There are some obvious cases of nonuniqueness: If a player is dealt four deuces, he may hold or discard the nonwild card with no effect. If he is dealt two pair without deuces, it is usually optimal to discard one of the pairs—it does not matter which one. Finally, it is frequently the case that a hand with two four-card straights has two optimal one-card discards. For example, with A♣-K♣-Q♢-J-♡-9♣, we can discard the ace or the 9 with no effect. In all of these examples, the payout distribution is unaffected by the choice of optimal strategy. But that is not true in general.

Consider the hand A♣-K♢-Q♢-J♡-9♢. As in the preceding example, there are two optimal one-card discards (the ace and the 9), but here there is a third optimal strategy, namely to hold the three-card straight flush. To confirm that all three strategies are optimal, it suffices to evaluate the expected payout in each case. If we hold either of the four-card straights, it is

$$E[R] = 2\left(\frac{8}{47}\right) + 0\left(\frac{39}{47}\right) = \frac{16}{47} \approx 0.340426. \tag{17.21}$$

If we hold the three-card straight flush, it is

$$\begin{aligned} E[R] &= 9\,\frac{\binom{6}{2}}{\binom{47}{2}} + 2\,\frac{\binom{4}{0}\left[\binom{9}{2}-1\right] + \binom{4}{1}\binom{7}{1} + \binom{4}{0}\left[\binom{4}{1}\binom{3}{1}-1\right] + \binom{4}{1}\binom{5}{1}}{\binom{47}{2}} \\ &\quad + 1\,\frac{\binom{4}{0}\binom{3}{1}\binom{3}{2} + \binom{4}{1}\binom{3}{1}\binom{3}{1}}{\binom{47}{2}} = \frac{368}{1{,}081} = \frac{16}{47} \approx 0.340426. \end{aligned} \tag{17.22}$$

The four terms in the numerator of the second fraction correspond to (a) flushes without deuces, (b) flushes with one deuce, (c) straights without deuces, and (d) straights with one deuce.

This is an example of an equivalence class for which the optimal strategy is not only nonunique but is what we will call *essentially nonunique*, in that the choice of strategy affects the payout distribution. There are 572 equivalence classes with this property, 286 of size 12 and 286 of size 24. Notice that the variance corresponding to (17.21) is smaller than that corresponding to (17.22). The optimal player who discards only one card in each such situation is using the *minimum-variance optimal strategy*, whereas the optimal player who discards two cards in each such situation is using the *maximum-variance optimal strategy*.

Thus, the payout distribution for Deuces Wild played according to the minimum-variance optimal strategy is uniquely determined, and we display it in Table 17.11. The same is true of the payout distribution for Deuces Wild played according to the maximum-variance optimal strategy, but we do not display it.

Here it is worth giving exact results. A common denominator (not the least one) is (17.11) on p. 552. The least common denominator is this number divided by 12.

It follows that the mean payout under optimal play is

$$\frac{1{,}673{,}759{,}500{,}036}{1{,}661{,}102{,}543{,}100} = \frac{32{,}187{,}682{,}693}{31{,}944{,}279{,}675} \approx 1.007619612039. \tag{17.23}$$

We have discovered something remarkable: Deuces Wild is a rare example of a casino game that offers positive expected profit to the knowledgeable player (about 3/4 of one percent). The variance of the payout, under the minimum-variance optimal strategy, is 25.834618052354.

Table 17.11 The distribution of the payout R from a one-unit bet on the video poker game Deuces Wild. Assumes maximum-coin bet and the minimum-variance optimal drawing strategy.

result	R	probability	probability $\times \binom{52}{5}\binom{47}{5}5/12$
natural royal flush	800	.000 022 083 864	36,683,563
four deuces	200	.000 203 703 199	338,371,902
wild royal flush	25	.001 795 843 261	2,983,079,808
five of a kind	15	.003 201 603 965	5,318,192,488
straight flush	9	.004 119 878 191	6,843,540,140
four of a kind	5	.064 938 165 916	107,868,952,548
full house	3	.021 229 137 790	35,263,774,770
flush *or* straight	2	.073 145 116 685	121,501,539,340
three of a kind	1	.284 544 359 823	472,657,359,726
other	0	.546 800 107 307	908,291,048,815
total		1.000 000 000 000	1,661,102,543,100

There remains an important issue that has not yet been addressed. What exactly is the optimal strategy at Deuces Wild? Our computer program provides one possible answer: specific instructions for each of the 102,359 equivalence classes. However, what we need is something simpler, a strategy that can actually be memorized. The usual approach is to construct a so-called *hand-rank table*. Each of the various types of holdings is ranked according to its conditional expectation (which varies slightly with the cards discarded). One then simply finds the highest-ranked hand in the table that is applicable to the hand in question. Only relatively recently has a hand-rank table been

Table 17.12 The (minimum-variance) optimal strategy for Deuces Wild. Count the number of deuces and choose the corresponding strategy ranked highest in the table, holding only the cards listed in that strategy. If none applies, hold only the deuces. Abbreviations: RF = royal flush, 4D = four deuces, 5K = five of a kind, SF = straight flush, 4K = four of a kind, FH = full house, F = flush, S = straight, 3K = three of a kind, 2P = two pair, 1P = one pair. n-RF, n-SF, n-F, and n-S refer to n-card hands that have the potential to become RF, SF, F, and S, respectively. 4-4D, 3-3K, 4-2P, and 2-1P have a slightly different meaning: For example, 3-3K is a 3-card three of a kind, i.e., the potential is already realized. A, K, Q, J, T denote ace, king, queen, jack, and ten. If two or more cards are italicized, that indicates that they are of the same suit. s is the number of straights, disregarding suits, that can be made from the hand, *excluding 2-low straights*. fp, sp, 8p, etc. denote flush penalty, straight penalty, 8 straight penalty, etc. Finally, unp. S pot. stands for unpenalized straight potential.

rank	description	rank	description
four deuces		no deuces	
1	5-4D	1	5-RF
		2	4-RF
three deuces		3	5-SF
1	5-RF	4	4-4K
2	5-5K: 222AA–222TT	5	5-FH, 5-F, or 5-S
		6	3-3K
two deuces		7	4-SF: $s = 2$
1	5-RF, 5-5K, or 5-SF	8	3-RF: *QJT*, no Kp
2	4-RF or 4-4K	9	4-SF: $s = 1$
3	4-SF: $s = 4$	10	3-RF
		11	2-1P
one deuce		12	4-F or 4-S: $s = 2$
1	5-RF, 5-5K, 5-SF, or 5-FH	13	3-SF: $s \geq 2$
2	4-RF or 4-4K	14	3-SF: *JT7*, Ap+Kp or Qp but not Qp+8p
3	4-SF: $s = 3$		
4	5-F or 5-S	15	2-RF: *JT*
5	3-3K	16	2-RF: *QJ* or *QT*, no sp
6	4-SF: $s \leq 2$	17	3-SF: $s = 1$, except A-low, no sp
7	3-RF: Q-high or J-high	18	2-RF: *QJ* or *QT*, no fp, unp. S pot.
8	3-RF: K-high	19	2-RF: *QT* if *QT*876
9	3-SF: $s = 4$	20	4-S: $s = 1$, except A-low
10	3-RF: A-high, no fp or sp*	21	3-SF: $s = 1$, except A-low
		22	2-RF: *QJ* or *QT*
		23	2-RF: *KQ*, *KJ*, or *KT*, no fp or sp*

*there are exceptions (see Table 17.17)

found for Deuces Wild that reproduces the optimal strategy precisely, and that table is presented as Table 17.12.

Let us consider the example, K♣-Q♢-J♢-T♢-8♢, discussed near the beginning of this section. Recall that we identified the three most promising strategies: (a) hold the four-card open-ended straight, (b) hold the four-card one-gap straight flush, and (c) hold the three-card royal flush.

Under no deuces, (a) is ranked 12th, (b) is ranked ninth, and (c) is ranked tenth. Holding the four-card one-gap straight flush ranks highest and is therefore the correct strategy, as we have already seen.

Finally, we reconsider the closely related example Q♢-J♢-T♢-8♢-7♣. The two best strategies are (b) and (c) from the preceding example.

(b) is still ranked ninth, but (c) is now ranked eighth. Holding the three-card royal flush ranks highest and is therefore the correct strategy, as we have already seen. This emphasizes the fact the our hand-rank table is sensitive enough to take penalty cards into account.

17.3 Problems

17.1. *A Jacks or Better strategy decision.* In Jacks or Better it is optimal to hold K♣-Q♣-J♣-T♣-9♣ for a guaranteed payout of 50 units rather than to discard the 9 and draw for the royal flush. How large would the payout on a royal flush have to be, all other things being equal, to make it optimal to discard the 9?

17.2. *A Deuces Wild strategy decision.* In Deuces Wild it is optimal to hold A♣-K♣-Q♣-J♣-2♢ for a guaranteed payout of 25 units rather than to discard the deuce and draw for the natural royal flush. How large would the payout on a natural royal flush have to be, all other things being equal, to make it optimal to discard the deuce?

17.3. *Jacks or Better practice hands.* For each initial hand listed in Table 17.13, (a) guess the optimal strategy, (b) use Table 17.5 on p. 554 to determine the optimal strategy, and (c) compute the conditional expectations of all promising strategies, using either combinatorics or a computer program.

17.4. *Deuces Wild practice hands.* For each initial hand listed in Table 17.14, (a) guess the optimal strategy, (b) use Table 17.12 on p. 565 to determine the optimal strategy, and (c) compute the conditional expectations of all promising strategies, using either combinatorics or a computer program.

17.5. *Five of a kind with three deuces.* In Deuces Wild we showed in (17.17) and (17.18) on p. 560 that it is optimal to hold all five cards if dealt three deuces and two tens but not if dealt three deuces and two 9s.

(a) Extend this analysis to all three-deuce five-of-a-kind hands, of which there are 12 equivalence classes. Use combinatorial analysis.

Table 17.13 Some Jacks or Better practice hands.

1.	A♢-Q♢-J♢-T♢-6♢	11.	J♡-A♣-9♢-K♣-Q♡
2.	K♣-Q♢-J♢-T♢-9♢	12.	J♡-A♣-8♡-K♣-Q♡
3.	Q♣-Q♢-J♣-9♣-8♣	13.	A♡-Q♢-T♢-9♣-8♢
4.	K♢-Q♢-J♢-T♢-9♢	14.	A♡-J♣-Q♠-T♠-8♠
5.	A♣-Q♣-J♣-9♣-7♢	15.	7♡-A♡-Q♡-K♠-T♡
6.	K♣-J-♣-T♣-9♣-9♢	16.	K♡-T♡-3♣-9♣-5♡
7.	9♢-8♣-7♠-6♡-6♢	17.	T♡-K♡-8♡-9♢-J♠
8.	K♠-Q♣-J♣-8♣-7♣	18.	A♡-K♡-T♡-5♡-K♠
9.	A♢-K♣-Q♡-J♣-6♠	19.	T♣-K♢-6♣-9♢-J♣
10.	K♢-Q♠-J♢-7♠-5♠	20.	9♠-J♣-3♡-4♡-7♡

Table 17.14 Some Deuces Wild practice hands.

1.	K♣-Q♣-J♣-T♣-9♣	11.	7♣-K♢-J♣-A♢-T♣
2.	K♣-Q♣-J♣-9♣-9♢	12.	8♡-Q♠-J♡-T♠-7♡
3.	K♢-8♢-5♢-3♢-3♡	13.	3♠-2♡-K♠-7♢-A♠
4.	9♠-8♠-2♣-2♢-2♡	14.	9♢-T♡-5♢-Q♡-8♢
5.	T♡-9♡-3♡-2♡-2♠	15.	6♢-K♣-7♠-Q♣-8♢
6.	A♣-K♣-K♢-T♣-9♣	16.	7♢-2♣-A♡-6♢-J♡
7.	7♣-6♡-5♡-2♢-2♠	17.	T♡-Q♡-7♡-A♣-K♢
8.	A♡-A♠-K♡-K♠-Q♡	18.	T♡-8♠-7♡-Q♣-J♡
9.	K♣-Q♢-J♢-T♢-9♢	19.	J♠-8♠-A♡-Q♠-T♠
10.	A♠-J♠-T♠-9♠-2♡	20.	5♡-A♣-3♡-4♣-6♡

(b) In the cases of part (a) where it is optimal to draw, evaluate the variance of the payout. In these cases, departing from optimal play reduces the expected payout very slightly but reduces the variance of the payout dramatically.

17.6. *Variance of the payout under maximum-variance optimal strategy.* In Deuces Wild we found that the variance of the payout is approximately 25.834618052 under the minimum-variance optimal strategy. Find the corresponding figure for the maximum-variance optimal strategy.

17.7. *Distribution of the number of cards held.*

(a) Assuming the optimal strategy at Jacks or Better (and standing pat with four of a kind), find the joint distribution of the payout and the number of cards held. How many of the 60 joint probabilities are 0? Find the marginal distribution of the number of cards held.

(b) Do the same for Deuces Wild, assuming the minimum-variance optimal strategy (and standing pat with four deuces).

17.8. *Deuces Wild conditional probabilities.*

(a) Find the conditional probabilities of the 10 payouts in Deuces Wild (assuming the minimum-variance optimal strategy), given the number of deuces in the hand before the draw. The unconditional probabilities are displayed in Table 17.11 on p. 564. How many of these 50 conditional probabilities are 0?

(b) Find the conditional expected payouts in Deuces Wild, given the number of deuces in the hand before the draw.

17.9. *A dubious notion of optimality.*

(a) In Jacks or Better, suppose the goal is to maximize the probability of obtaining a royal flush. In other words, instead of choosing the strategy that maximizes the conditional expected payout, we choose one that maximizes the conditional probability of a royal flush. (Alternatively, we could adjust the payoff odds so that only a royal flush has a positive payout.) What is this maximum probability? Use combinatorial analysis.

(b) Is the answer the same for Deuces Wild and a natural royal flush?

17.10. *n-play video poker*. Fix an integer $n \geq 1$. In n-play video poker, the player bets n units. He then receives five cards face up on the screen, with each of the $\binom{52}{5}$ possible hands equally likely. For each card, the player must then decide whether to hold or discard that card. Thus, there are 2^5 ways to play the hand. If he discards k cards, then the following occurs n times with the results conditionally independent: He is dealt k new cards, with each of the $\binom{47}{k}$ possibilities equally likely. The player then receives his payout, which depends on the payout schedule and assumes one unit bet on each of the n hands.

(a) Argue that the optimal strategy does not depend on n.

(b) Show that the variance of the *payout per unit bet* is decreasing in n. More precisely, let $X_1, \ldots, X_n$ be the payouts from the n plays and let Y denote the initial hand. Then $X_1, \ldots, X_n$ are conditionally i.i.d. given Y. Use the conditioning law for variances (Theorem 2.2.13 on p. 87) to show that

$$\text{Var}\left(\frac{X_1 + \cdots + X_n}{n}\right) = \text{Var}(X_1) - \left(1 - \frac{1}{n}\right)\text{E}[\text{Var}(X_1|Y)]. \tag{17.24}$$

17.11. *Probability of a royal at n-play Jacks or Better vs. at n independent games of Jacks or Better*. Fix an integer $n \geq 2$. Assuming the optimal strategy, which is more likely, at least one royal flush at one game of n-play Jacks or Better (Problem 17.10) or at least one royal flush at n *independent* games of (1-play) Jacks or Better?

17.12. *Jacks or Better: A mathematical simplification.* In Jacks or Better, assume for simplicity that when k cards are discarded, the k new cards are chosen at random without replacement from the $47 + k$ cards not held rather than from the 47 cards not seen. (To put it another way, we assume that the discards are shuffled back into the deck before their replacements are chosen.)

(a) Determine the optimal strategy under this simplification.

(b) Determine the expected payout using this simplified strategy in the real game.

Hint: (a) Let H denote the set of cards held. Evaluate the expected payout E_H for each H, starting with $H = \varnothing$, then $|H| = 1$ (52 cases), then $|H| = 2$ (1,326 cases), and so on. Keep track only of those E_H for which $\mathrm{E}_H \geq \mathrm{E}_G$ whenever $G \subset H$.

17.13. *Double Bonus video poker*. Double Bonus Poker is played with a standard 52-card deck and there are no wild cards. The payout schedule is shown in Table 17.15. Find the expected payout under optimal play.

Table 17.15 Double Bonus Poker payoff odds and pre-draw frequencies.

rank	payoff odds	number of ways
royal flush	800 for 1	4
straight flush	50 for 1	36
four aces	160 for 1	48
four 2s, 3s, or 4s	80 for 1	144
four of a kind (others)	50 for 1	432
full house	10 for 1	3,744
flush	7 for 1	5,108
straight	5 for 1	10,200
three of a kind	3 for 1	54,912
two pair	1 for 1	123,552
pair of jacks or better	1 for 1	337,920
other	0 for 1	2,062,860
total		2,598,960

17.14. *Joker Wild video poker*. Joker Wild adds a joker, which acts as a wild card, to the standard 52-card deck. The payout schedule is shown in Table 17.16.

(a) Confirm the pre-draw frequencies shown in the table.

(b) Define an equivalence relation on the set of all five-card hands, analogous to the one in Section 17.2, and determine the number of equivalence classes.

(c) Find the expected payout under optimal play.

Table 17.16 Joker Wild payoff odds and pre-draw frequencies.

rank	payoff odds	number of ways without a joker	number of ways with a joker
royal flush (natural)	800 for 1	4	0
five of a kind	200 for 1	0	13
royal flush (joker)	100 for 1	0	20
straight flush	50 for 1	36	144
four of a kind	20 for 1	624	2,496
full house	7 for 1	3,744	2,808
flush	5 for 1	5,108	2,696
straight	3 for 1	10,200	10,332
three of a kind	2 for 1	54,912	82,368
two pair	1 for 1	123,552	0
pair of aces or kings	1 for 1	168,960	93,996
other	0 for 1	2,231,820	75,852
total		2,598,960	270,725

17.4 Notes

Video poker became firmly established with the introduction by SIRCOMA (Si Redd's [1911–2003] Coin Machines, which evolved into International Game Technology, or IGT) of "Draw Poker" in 1979. See Fey (2002, p. 217) for a picture of this machine, whose pay table was identical to that for full-pay Jacks or Better, except that the 1 for 1 payout on a pair of jacks or better was absent. As explained by Paymar (2004, p. 10), the manufacturer had not yet figured out how to evaluate the game's expected payback, so they initially played it safe. However, there was little interest by the gambling public in such an unfair game, so the computer chip was modified and a placard was attached to the machine saying "Bet returned on a pair of jacks or better." This was the origin of today's full-pay Jacks or Better. However, there were several predecessors, so the year of video poker's debut is arguable.

Bally Manufacturing Corporation introduced a video poker machine in 1976, according to Weber and Scruggs (1992), though it was not mentioned by Jones (1978). Dale Electronics introduced an electronic machine called "Poker Matic" in 1970, but it did not have a video monitor, so strictly speaking it was not video poker. But if the video aspect is inessential to the nature of the game, then so too is the electronic aspect, and we have to go back to 1901, when Charles August Fey [1862–1944] introduced the first poker machine with a draw feature (Fey 2002, p. 76). There were 10 cards on each of five reels, so only $(10)^5 = 100{,}000$ hands, not 2,598,960, were possible. It is this issue, we believe, not the lack of a video monitor, that disqualifies the Fey machine as the first video poker machine.

Deuces Wild came later and remains one of the more popular forms of video poker. Today there are dozens of variations on the original game. Unfortunately, according to Bob Dancer (personal communication, 2009), full-pay Deuces Wild "is down to perhaps 100 machines in Las Vegas—and five years ago there were more than 1,000."

Because of the complications of video poker, optimal strategies were not immediately forthcoming. Early attempts used computer simulation rather than exact computation. Some of the earliest authors include Wong (1988), Frome (1989), and Weber and Scruggs (1992).

For a group-theoretic approach to counting the 134,459 equivalence classes of pre-draw hands at Jacks or Better, see Alspach (2007).

The Deuces Wild pre-draw frequencies of Tables 17.7 and 17.8 are well known. However, Goren's *Hoyle* (1961, p. 110) has them wrong, giving for example 4,072 ways to be dealt a nonroyal straight flush instead of the correct 2,068. Russell (1983, pp. 6, 83) and Percy (1988, p. 16) have the same errors.

Exactly optimal strategies are available from only a few sources. Our Tables 17.5 and 17.12 are adapted from Dancer and Daily (2004, Chapter 6; 2003, Chapter 6). The asterisks in Table 17.12 hide a number of exceptions, for which the reader is referred to Table 17.17 below. Paymar (2004) has made the case that the mathematically optimal strategy is less efficient in practice than a simplified version of it that he called "precision play."

Marshall (2006) published the complete list of the 134,459 equivalence classes of hands in Deuces Wild. (He regarded deuces as having suits, unlike our approach.) This required some 357 pages, seven columns per page, 55 hands per column. For each equivalence class he provided the (more precisely, an) optimal play. For example, the equivalence class containing A♣-K♢-Q♢-J♡-9♢ is listed as KQ9 · A · J with the cards to be held underlined. It appears that his strategy is the minimum-variance one.

As for exactly optimal strategies for Jacks or Better, we quote the following summary of the situation as of the mid-1990s (Gordon 2000):

> Paymar (1994, p. 9) claims that the 99.54 figure he reported is an absolute maximum. That claim is surprising since Frome's 99.6 had previously been published and Paymar was aware of it. No basis for that claim is given other than unidentified "independent analyses." Such upper bounds are difficult to establish in much simpler analyses and are impossible for a complicated analysis like this. Frome's counter-example is sufficient to render the claim invalid.

See (17.12) for the exact figure. Frome's number was simply inaccurate. Gordon (2000) went on to claim a 99.75 percent expected payout, which is even less accurate.

The exact optimal strategy for Jacks or Better was said by Paymar (1998, p. 45) to have been published, though he did not say where or by whom. As far as we know, the first exact evaluation of the payout distribution for Deuces Wild was obtained by Jensen (2001), and his numbers were used by Ethier and Khoshnevisan (2002). However, they differ slightly from those of Table 17.11 because he found the maximum-variance optimal strategy, apparently unaware of the nonuniqueness issue.

Most of the examples in Problems 17.3 and 17.4 are from Dancer and Daily (2004, 2003), as is Problem 17.5 (2003, Appendix A). Double Bonus video poker (Problem 17.13) and Joker Wild video poker (Problem 17.14) have been studied by the same authors.

Table 17.17 Exceptions to Table 17.12. From Dancer and Daily (2003, Appendices B and C); two minor mistakes have been corrected.

Exceptions to one-deuce rule 10: 3-RF: A-high, no fp or sp
hold 3-RF, even though it is penalized:
3-RF: AJT7 when T7 are unsuited with each other and with AJ
3-RF: AJT7 when J7 are unsuited with each other and with AT
hold only the deuce, even though 3-RF is unpenalized:
3-RF: AK and 93, *83*, 73, 64, 63, 54, 53, or 43
3-RF: AQ and 63, *54*, 53, or 43
3-RF: AJ and *53* or 43
3-RF: AT and *43*
Exceptions to no-deuce rule 23: 2-RF: KQ, KJ, or KT, no fp or sp
hold 2-RF, even though it is penalized:
2-RF: KQ9 and 87 or 76
2-RF: KQ9 and 86 when nonKQ cards are of different suits or *96*
2-RF: KQ9 and 85 or 75 when nonKQ cards are of different suits
2-RF: AKJ and 87 when nonKJ cards are of different suits
2-RF: KJ9 and 86, 85, 76, or 75
2-RF: KJ9 and 74 or 65 when nonKJ cards are of different suits
2-RF: AKT and 87 when nonKT cards are not all of the same suit
2-RF: AKT and 86 or 76 when nonKT cards are of different suits
2-RF: KT9 and 85, 75, or 65
2-RF: KT9 and 84 or 74 when nonKT cards are not all of the same suit
2-RF: KT9 and 73 or 64 when nonKT cards are of different suits
hold nothing, even though 2-RF is unpenalized:
2-RF: KQ and 743 when *74* or *43*
2-RF: KQ and 653 when two of the nonKQ cards are suited
2-RF: KQ and 643 or 543
2-RF: KJ and 643 when *64* or *43*
2-RF: KJ and 543

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