Probability and its Applications

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Stewart N. Ethier

The Doctrine of Chances

Probabilistic Aspects of Gambling

Springer

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Preface

> I have found many thousands more readers than I ever looked for. I have no right to say to these, You shall not find fault with my art, or fall asleep over my pages; but I ask you to believe that this person writing strives to tell the truth. If there is not that, there is nothing.
>
> William Makepeace Thackeray, *The History of Pendennis*

This is a monograph/textbook on the probabilistic aspects of gambling, intended for those already familiar with probability at the post-calculus, pre-measure-theory level.

Gambling motivated much of the early development of probability theory (David 1962).[1] Indeed, some of the earliest works on probability include Girolamo Cardano's [1501–1576] *Liber de Ludo Aleae* (*The Book on Games of Chance*, written c. 1565, published 1663), Christiaan Huygens's [1629–1695] "De ratiociniis in ludo aleae" ("On reckoning in games of chance," 1657), Jacob Bernoulli's [1654–1705] *Ars Conjectandi* (*The Art of Conjecturing*, written c. 1690, published 1713), Pierre Rémond de Montmort's [1678–1719] *Essay d'analyse sur les jeux de hasard* (*Analytical Essay on Games of Chance*, 1708, 1713), and Abraham De Moivre's [1667–1754] *The Doctrine of Chances* (1718, 1738, 1756). Gambling also had a major influence on 20th-century probability theory, as it provided the motivation for the concept of a martingale.

Thus, gambling has contributed to probability theory. Conversely, probability theory has contributed much to gambling, from the gambler's ruin formula of Blaise Pascal [1623–1662] to the optimality of bold play due to Lester E. Dubins [1920–2010] and Leonard J. Savage [1917–1971]; from the solution of le her due to Charles Waldegrave to the solution of chemin de fer due to John G. Kemeny [1926–1992] and J. Laurie Snell [1925–]; from the duration-of-play formula of Joseph-Louis Lagrange [1736–1813] to the optimal proportional betting strategy of John L. Kelly, Jr. [1923–1965]; and from

[1] See Maistrov (1974, Chapter 1, Section 2) for a different point of view.

the first evaluation of the banker's advantage at trente et quarante due to Siméon-Denis Poisson [1781–1840] to the first published card-counting system at twenty-one due to Edward O. Thorp [1932–]. Topics such as these are the principal focus of this book.

Is gambling a subject worthy of academic study? Let us quote an authority from the 18th century on this question. In the preface to *The Doctrine of Chances*, De Moivre (1718, p. iii) wrote,

> Another use to be made of this Doctrine of Chances is, that it may serve in Conjunction with the other parts of the Mathematicks, as a fit introduction to the Art of Reasoning; it being known by experience that nothing can contribute more to the attaining of that Art, than the consideration of a long Train of Consequences, rightly deduced from undoubted Principles, of which this Book affords many Examples.

We also quote a 20th-century authority on the same question. In *Le jeu, la chance et le hasard*, Louis Bachelier [1870–1946] (1914, p. 6) wrote,[2]

> It is almost always gambling that enables one to form a fairly clear idea of a manifestation of chance; it is gambling that gave birth to the calculus of probability; it is to gambling that this calculus owes its first faltering utterances and its most recent developments; it is gambling that allows us to conceive of this calculus in the most general way; it is, therefore, gambling that one must strive to understand, but one should understand it in a philosophic sense, free from all vulgar ideas.

Certainly, there are other applications of probability theory on which courses of study could be based, and some of them (e.g., actuarial science, financial engineering) may offer better career prospects than does gambling! But gambling is one of the only applications in which the probabilistic models are often *exactly* correct.[3] This is due to the fundamental simplicity of the nature of the randomness in games of chance. This simplicity translates into an elegance that few other applications enjoy.

The book consists of two parts. Part I ("Theory") begins with a review of probability, then turns to several probability topics that are often not covered in a first course (conditional expectation, martingales, and Markov chains), then briefly considers game theory, and finally concludes with various gambling topics (house advantage, gambler's ruin, betting systems, bold play, optimal proportional play, and card theory). Part II ("Applications") discusses a variety of casino games, including six games in which successive coups are independent (slot machines, roulette, keno, craps, house-banked poker, and video poker) and four games with dependence among coups (faro, baccarat, trente et quarante, and twenty-one). Within each group, chapters are ordered according to difficulty but are largely independent of one another and can be read in any order. We conclude with a discussion of poker, which is in a class by itself.

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[2] Translation from Dubins and Savage (1976).

[3] Here, and throughout the book (perhaps with the exception of Section 13.2), we model the ideal, or *benchmark*, game, the game as it is intended to be played by the manufacturer of the dice, cards, wheels, machines, etc.

The only contemporary book with comparable content and prerequisites is Richard A. Epstein's [1927–] *The Theory of Gambling and Statistical Logic* (1967, 1977, 2009). Epstein's book is fun to read but is not entirely suitable as a textbook: It is a compendium of results, often without derivations, and there are few problems or exercises to reinforce the reader's understanding. Our aim was not only to supply the missing material but to provide more-self-contained and more-comprehensive coverage of the principal topics. We have tried to do this without sacrificing the "fun to read" factor.

Although there is enough material here for a two-semester course, the book could be used for a one-semester course, either by covering some subset of the chapters thoroughly (perhaps assigning other chapters as individual projects) or by covering every chapter less than thoroughly. In an NSF-sponsored Research Experience for Undergraduates (REU) summer program at the University of Utah in 2005, we adopted the latter approach using a preliminary draft of the book. Fred M. Hoppe, in a course titled "Probability and Games of Chance" at McMaster University in spring 2009, adopted the former approach, covering Chapters 1, 2, 17, 3, 15, and 6 in that order.

The book is not intended solely for American and Canadian readers. Money is measured in units, not dollars, and European games, such as chemin de fer and trente et quarante, are studied. This is appropriate, inasmuch as France is not only the birthplace of probability theory but also that of roulette, faro, baccarat, trente et quarante, and twenty-one.

With few exceptions, all random variables in the book are discrete.[4] This allows us to provide a mathematically rigorous treatment, while avoiding the need for measure theory except for occasional references to the Appendix. Each chapter contains a collection of problems that range from straightforward to challenging. Some require computing. Answers, but not solutions, will be provided at the author's web page (http://www.math.utah.edu/~ethier/). While we have not hesitated to use computing in the text (in fact, it is a necessity in studying such topics as video poker, twenty-one, and Texas hold'em), we have avoided the use of computer simulation, which seems to us outside the spirit of the subject. Each chapter also contains a set of historical notes, in which credit is assigned wherever possible and to the best of our knowledge. This has necessitated a lengthy bibliography. In many cases we simply do not know who originated a particular idea, so a lack of attribution should not be interpreted as a claim of originality. We frequently refer to the generic gambler, bettor, player, dealer, etc. with the personal pronoun "he," which has the old-fashioned interpretation of, but is much less awkward than, "he or she."

A year or two ago (2008) was the tercentenary of the publication of the first edition of Montmort's *Analytical Essay on Games of Chance*, which can

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[4] The only exceptions are nondiscrete limits of sequences of discrete random variables. These may occur, for example, in the martingale convergence theorem.

be regarded as the first published full-length book on probability theory.[5] As Todhunter (1865, Article 136) said of Montmort,

> In 1708 he published his work on Chances, where with the courage of Columbus he revealed a new world to mathematicians.

A decade later De Moivre published his equally groundbreaking work, *The Doctrine of Chances*. Either title would be suitable for the present book; we have chosen the latter because it sounds a little less intimidating.

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Finally, I am especially grateful to my wife, Kyoko, for her patience throughout this lengthy project.

Dedication: The book is dedicated to the memory of gambling historian Russell T. Barnhart [1926–2003] and twenty-one theorist Peter A. Griffin [1937–1998], whom I met in 1984 and 1981, respectively. Their correspondence about gambling matters over the years fills several thick folders (neither used e-mail), and their influence on the book is substantial.

Salt Lake City, December 2009 Stewart N. Ethier

[5] Cardano's *Liber de Ludo Aleae* comprises only 15 (dense) pages of his *Opera omnia* and Huygens's "De ratiociniis in ludo aleae" comprises only 18 pages of van Schooten's *Exercitationum Mathematicarum*.

Contents

List of Notation

symbol	meaning	page of first use
♠	end of proof or end of example[1]	6
$:=$	equals by definition ($=:$ is also used)	3
$\equiv$	is identically equal to, or is congruent to	59
$\approx$	is approximately equal to	9
$\sim$	is asymptotic to ($a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$)	48
$\mathbf{N}$	the set of positive integers	58
$\mathbf{Z}_+$	the set of nonnegative integers	91
$\mathbf{Z}$	the set of integers	58
$\mathbf{Q}$	the set of rational numbers	108
$\mathbf{R}$	the set of real numbers	21
$\lvert x\rvert$	absolute value of real x; modulus of complex x	6
x^+	nonnegative part of the real number x ($:= \max(x, 0)$)	22
x^-	nonpositive part of the real number x ($:= -\min(x, 0)$)	64
$\boldsymbol{x} \cdot \boldsymbol{y}$	inner product of $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^d$	34
$\lvert\boldsymbol{x}\rvert$	Euclidean norm of vector $\boldsymbol{x} \in \boldsymbol{R}^d$ ($:= (\boldsymbol{x} \cdot \boldsymbol{x})^{1/2}$)	177
$\boldsymbol{A}^\top$	transpose of matrix (or vector) $\boldsymbol{A}$	176
$0.4\overline{92}$	$0.4929292\cdots$ (repeating decimal expansion)	19
$\mathrm{sgn}(x)$	sign of x ($:= 1, 0, -1$ if $x > 0,\ = 0,\ < 0$)	170
$\lvert A\rvert$	cardinality of (number of elements of) the finite set A	3
$x \in A$	x is an element of A	3
$A \subset B$	A is a (not necessarily proper) subset of B	3
$B \supset A$	equivalent to $A \subset B$	13

[1]In discussions of card games, the symbol ♠ signifies a spade.

symbol	meaning	page of first use
$A \cup B$	union of A and B	11
$A \cap B$	intersection of A and B	11
A^c	complement of A	11
$A - B$	set-theoretic difference ($:= A \cap B^c$)	11
$A \times B$	cartesian product of A and B	15
A^n	n-fold cartesian product $A \times \cdots \times A$	25
1_A	indicator r.v. of event A or indicator function of set A	33
$x \vee y$	$\max(x, y)$	56
$x \wedge y$	$\min(x, y)$	6
$\ln x$	natural (base e) logarithm of x	30
$\log_2 x$	base-2 logarithm of x	48
$\lfloor x \rfloor$	the greatest integer less than or equal to x	39
$\lceil x \rceil$	the least integer greater than or equal to x	30
$\varnothing$	empty set	12
$(n)_k$	$:= n(n-1)\cdots(n-k+1)$ if $k \geq 1$, $(n)_0 := 1$	4
$n!$	n factorial ($:= (n)_n = n(n-1)\cdots 1$ if $n \geq 1$, $0! := 1$)	4
$\binom{n}{k}$	binomial coefficient n choose k ($:= (n)_k/k!$)	4
$\binom{n}{n_1,\ldots,n_r}$	multinomial coefficient ($:= n!/(n_1!\cdots n_r!)$)	5
$\delta_{i,j}$	Kronecker delta ($:= 1$ if $i = j$, $:= 0$ otherwise)	20
(a, b)	open interval $\{x : a < x < b\}$	22
$[a, b)$	half-open interval $\{x : a \leq x < b\}$	27
$(a, b]$	half-open interval $\{x : a < x \leq b\}$	140
$[a, b]$	closed interval $\{x : a \leq x \leq b\}$	21
$\xrightarrow{d}$	converges in distribution to	42
$N(0, 1)$	standard-normal distribution	42
$\Phi(x)$	standard-normal cumulative distribution function	42
$\phi(x)$	standard-normal density function ($:= \Phi'(x)$)	63
a.s.	almost surely	42
i.o.	infinitely often	43
i.i.d.	independent and identically distributed	43
g.c.d.	greatest common divisor	137

Positive, negative, increasing, and decreasing are used only in the strict sense. When the weak sense is intended, we use nonnegative, nonpositive, nondecreasing, and nonincreasing, respectively.