# A World Record in Atlantic City and the Length of the Shooter's Hand at Craps

S. N. ETHIER AND FRED M. HOPPE*

It was widely reported in the media that, on 23 May 2009, at the Borgata Hotel Casino & Spa in Atlantic City, Patricia DeMauro (spelled Demauro in some accounts), playing craps for only the second time, rolled the dice for four hours and 18 minutes, finally sevening out at the 154th roll. Initial estimates of the probability of a run at least this long (assuming fair dice and independent rolls) ranged from one chance in 3.5 billion [5] to one chance in 1.56 trillion [10]. Subsequent computations agreed on one chance in 5.6 (or 5.59) billion [2, 6, 9].

This established a new world record. The old record was held by the late Stanley Fujitake (118 rolls, 28 May 1989, California Hotel and Casino, Las Vegas) [1]. One might ask how reliable these numbers (118 and 154) are. In Mr. Fujitake's case, casino personnel replayed the surveillance videotape to confirm the number of rolls and the duration of time (three hours and six minutes). We imagine that the same happened in Ms. DeMauro's case.

There is also a report that Mr. Fujitake's record was broken earlier by a gentleman known only as The Captain (148 rolls, July 2005, Atlantic City) [8, Part 4]. However, this incident is not well documented (specifically, the exact date and casino name were not revealed) and was unknown to Borgata officials. In fact, a statistical argument has been offered [4, p. 480] suggesting that the story is apocryphal.

Our aim in this article is not simply to derive a more accurate probability, but to show that this apparently prosaic problem involves some interesting mathematics, including Markov chains, matrix theory, generating functions, and Galois theory.

## Background

Craps is played by rolling a pair of dice repeatedly. For most bets, only the sum of the numbers appearing on the two dice matters, and this sum has distribution

$$\pi_j := \frac{6 - |j - 7|}{36}, \qquad j = 2, 3, \ldots, 12. \tag{1}$$

The basic bet at craps is the *pass-line bet*, which is defined as follows. The first roll is the *come-out roll*. If 7 or 11 appear (a *natural*), the bettor wins. If 2, 3, or 12 appears (a *craps number*), the bettor loses. If a number belonging to

$$\mathscr{P} := \{4, 5, 6, 8, 9, 10\}$$

appears, that number becomes the *point*. The dice continue to be rolled until the point is repeated (or *made*), in which

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case the bettor wins, or 7 appears, in which case the bettor loses. The latter event is called a *seven out*. The first roll following a decision is a new come-out roll, beginning the process again.

A shooter is permitted to roll the dice until he or she sevens out. The sequence of rolls by the shooter is called the *shooter's hand*. Notice that the shooter's hand can contain winning 7s and losing decisions prior to the seven out. The *length* of the shooter's hand (i.e., the number of rolls) is a random variable we will denote by $L$. Our concern here is with

$$t(n) := \mathrm{P}(L \geq n), \qquad n \geq 1, \tag{2}$$

the tail of the distribution of $L$. For example, $t(154)$ is the probability of achieving a hand at least as long as that of Ms. DeMauro. As can be easily verified from (3), (6), or (9) below, $t(154) \approx 0.178\,882\,426 \times 10^{-9}$; to state it in the way preferred by the media, this amounts to one chance in 5.59 billion, approximately. The 1 in 3.5 billion figure came from a simulation that was not long enough. The 1 in 1.56 trillion figure came from $(1-\pi_7)^{154}$, which is the right answer to the wrong question.

## Two Methods

We know of two methods for evaluating the tail probabilities (2). The first is by recursion. As pointed out in [3], $t(1) = t(2) = 1$ and

$$t(n) = \Big(1 - \sum_{j \in \mathscr{P}} \pi_j\Big) t(n-1) + \sum_{j \in \mathscr{P}} \pi_j (1 - \pi_j - \pi_7)^{n-2} + \sum_{j \in \mathscr{P}} \pi_j \sum_{l=2}^{n-1} (1 - \pi_j - \pi_7)^{l-2} \pi_j t(n-l) \tag{3}$$

for each $n \geq 3$. Indeed, for the event that the shooter sevens out in no fewer than $n$ rolls to occur, consider the result of the initial come-out roll. If a natural or a craps number occurs, then, beginning with the next roll, the shooter must seven out in no fewer than $n-1$ rolls. If a point number occurs, then there are two possibilities. Either the point is still unresolved after $n-2$ additional rolls, or it is made at roll $l$ for some $l \in \{2, 3, \ldots, n-1\}$ and the shooter subsequently sevens out in no fewer than $n-l$ rolls.

The second method, first suggested, to the best of our knowledge, by Peter A. Griffin in 1987 (unpublished) and rediscovered several times since, is based on a Markov chain. The state space is

$$S := \{\text{co, p4-10, p5-9, p6-8, 7o}\} \equiv \{1, 2, 3, 4, 5\}, \tag{4}$$

whose five states represent the events that the shooter is coming out, has established the point 4 or 10, has established the point 5 or 9, has established the point 6 or 8, and has sevened out. The one-step transition matrix, which can be inferred from (1), is

$$\boldsymbol{P} := \frac{1}{36}\begin{pmatrix} 12 & 6 & 8 & 10 & 0 \\ 3 & 27 & 0 & 0 & 6 \\ 4 & 0 & 26 & 0 & 6 \\ 5 & 0 & 0 & 25 & 6 \\ 0 & 0 & 0 & 0 & 36 \end{pmatrix}. \tag{5}$$

The probability of sevening out in $n-1$ rolls or fewer is then just the probability that absorption in state 7o occurs by the $(n-1)$th step of the Markov chain, starting in state co. A marginal simplification results by considering the 4 by 4 principal submatrix $\boldsymbol{Q}$ of (5) corresponding to the transient states. Thus, we have

$$t(n) = 1 - (\boldsymbol{P}^{n-1})_{1,5} = \sum_{j=1}^{4} (\boldsymbol{Q}^{n-1})_{1,j}. \tag{6}$$

Clearly, (3) is not a closed-form expression, and we do not regard (6) as being in closed form either. Is there a closed-form expression for $t(n)$?

## Positivity of the Eigenvalues

We begin by showing that the eigenvalues of $\boldsymbol{Q}$ are positive. The determinant of

$$\boldsymbol{Q} - z\boldsymbol{I} = \frac{1}{36}\begin{pmatrix} 12-36z & 6 & 8 & 10 \\ 3 & 27-36z & 0 & 0 \\ 4 & 0 & 26-36z & 0 \\ 5 & 0 & 0 & 25-36z \end{pmatrix}$$

is unaltered by row operations. From the first row, subtract $6/(27-36z)$ times the second row, $8/(26-36z)$ times the third row, and $10/(25-36z)$ times the fourth row, cancelling the entries 6/36, 8/36, and 10/36 and making the (1,1) entry equal to 1/36 times

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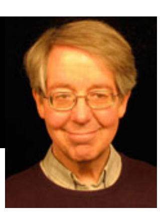

**S. N. ETHIER** is professor of mathematics at the University of Utah, and he specializes in applied probability. His book, *The Doctrine of Chances: Probabilistic Aspects of Gambling*, will be published this year by Springer.

Department of Mathematics
University of Utah
Salt Lake City, UT 84112
USA
e-mail: ethier@math.utah.edu

**FRED M. HOPPE** is professor of mathematics and statistics, and associate faculty of the Booth School of Engineering Practice, at McMaster University. At times, he has published on branching processes, population genetics, probability bounds, nuclear risk, and lotteries.

Department of Mathematics and Statistics
McMaster University
Hamilton, ON L8S 4K1
Canada
e-mail: hoppe@mcmaster.ca

$$12 - 36z - 3\frac{6}{27-36z} - 4\frac{8}{26-36z} - 5\frac{10}{25-36z}. \quad (7)$$

The determinant of $\boldsymbol{Q} - z\boldsymbol{I}$, and therefore the characteristic polynomial $q(z)$ of $\boldsymbol{Q}$, is then just the product of the diagonal entries in the transformed matrix, which is (7) multiplied by $(27-36z)(26-36z)(25-36z)/(36)^4$. Thus,

$$\begin{aligned} q(z) = {} & [(12-36z)(27-36z)(26-36z)(25-36z) \\ & - 18(26-36z)(25-36z) - 32(27-36z)(25-36z) \\ & - 50(27-36z)(26-36z)]/(36)^4. \end{aligned}$$

We find that $q(1)$, $q(27/36)$, $q(26/36)$, $q(25/36)$, and $q(0)$ alternate signs, and therefore the eigenvalues are positive and interlaced between the diagonal entries (ignoring the entry 12/36). More precisely, denoting the eigenvalues by $1 > e_1 > e_2 > e_3 > e_4 > 0$, we have

$$1 > e_1 > \frac{27}{36} > e_2 > \frac{26}{36} > e_3 > \frac{25}{36} > e_4 > 0.$$

The matrix $\boldsymbol{Q}$, which has the structure of an arrowhead matrix, is not symmetric, but is positive definite. A nonsymmetric matrix is positive definite if and only if its symmetric part is positive definite. This is easily seen to be the case for $\boldsymbol{Q}$ by applying the same type of row operations to the symmetric part $\boldsymbol{A} = \frac{1}{2}(\boldsymbol{Q} + \boldsymbol{Q}^\mathsf{T})$ to show that the eigenvalues of $\boldsymbol{A}$ interlace its diagonal elements (except 12/36), and hence are positive.

## A Closed-Form Expression

The eigenvalues of $\boldsymbol{Q}$ are the four roots of the quartic equation $q(z) = 0$ or

$$23328z^4 - 58320z^3 + 51534z^2 - 18321z + 1975 = 0, \quad (8)$$

while $\boldsymbol{P}$ has an additional eigenvalue, 1, the spectral radius. We can use the quartic formula (or *Mathematica*) to find these roots. We notice that the complex number

$$\alpha := \zeta^{1/3} + \frac{9829}{\zeta^{1/3}},$$

where

$$\zeta := -710369 + 18i\sqrt{1373296647},$$

appears three times in each root. Fortunately, $\alpha$ is positive, as we see by writing $\zeta$ in polar form, that is, $\zeta = re^{i\theta}$. We obtain

$$\alpha = 2\sqrt{9829}\cos\left[\frac{1}{3}\cos^{-1}\left(-\frac{710369}{9829\sqrt{9829}}\right)\right].$$

The four eigenvalues of $\boldsymbol{Q}$ can be expressed as

$$\begin{aligned} e_1 &:= e(1,1), \\ e_2 &:= e(1,-1), \\ e_3 &:= e(-1,1), \\ e_4 &:= e(-1,-1), \end{aligned}$$

where

$$\begin{aligned} e(u,v) := {} & \frac{5}{8} + \frac{u}{72}\sqrt{\frac{349+\alpha}{3}} \\ & + \frac{v}{72}\sqrt{\frac{698-\alpha}{3} - 2136u\sqrt{\frac{3}{349+\alpha}}}. \end{aligned}$$

Next we need to find right eigenvectors corresponding to the five eigenvalues of $\boldsymbol{P}$. Fortunately, these eigenvectors can be expressed in terms of the eigenvalues. Indeed, with $\boldsymbol{r}(x)$ defined to be the vector-valued function

$$\begin{pmatrix} -5 + (1/5)x \\ -175 + (581/15)x - (21/10)x^2 + (1/30)x^3 \\ 275/2 - (1199/40)x + (8/5)x^2 - (1/40)x^3 \\ 1 \\ 0 \end{pmatrix}$$

we find that right eigenvectors corresponding to eigenvalues 1, $e_1$, $e_2$, $e_3$, $e_4$ are

$$(1,1,1,1,1)^\mathsf{T},\ \boldsymbol{r}(36e_1),\ \boldsymbol{r}(36e_2),\ \boldsymbol{r}(36e_3),\ \boldsymbol{r}(36e_4),$$

respectively. Letting $\boldsymbol{R}$ denote the matrix whose columns are these right eigenvectors and putting $\boldsymbol{L} := \boldsymbol{R}^{-1}$, the rows of which are left eigenvectors, we know by (6) and the spectral representation that

$$t(n) = 1 - \{\boldsymbol{R}\ \mathrm{diag}(1, e_1^{n-1}, e_2^{n-1}, e_3^{n-1}, e_4^{n-1})\boldsymbol{L}\}_{1,5}.$$

After much algebra (and with some help from *Mathematica*), we obtain

$$t(n) = c_1 e_1^{n-1} + c_2 e_2^{n-1} + c_3 e_3^{n-1} + c_4 e_4^{n-1}, \quad (9)$$

where the coefficients are defined in terms of the eigenvalues and the function

$$\begin{aligned} f(w,x,y,z) := {} & (-25+36w)\big[4835 - 5580(x+y+z) \\ & + 6480(xy+xz+yz) - 7776xyz\big]/ \\ & [38880(w-x)(w-y)(w-z)] \end{aligned}$$

as follows:

$$\begin{aligned} c_1 &:= f(e_1,e_2,e_3,e_4), \\ c_2 &:= f(e_2,e_3,e_4,e_1), \\ c_3 &:= f(e_3,e_4,e_1,e_2), \\ c_4 &:= f(e_4,e_1,e_2,e_3). \end{aligned}$$

Of course, (9) is our closed-form expression.

Incidentally, the fact that $t(1) = t(2) = 1$ implies that

$$c_1 + c_2 + c_3 + c_4 = 1 \quad (10)$$

and

$$c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 = 1.$$

In a sequence of independent Bernoulli trials, each with success probability $p$, the number of trials $X$ needed to achieve the first success has the *geometric distribution* with parameter $p$, and

$$P(X \geq n) = (1-p)^{n-1}, \qquad n \geq 1.$$

It follows that the distribution of $L$ is *a linear combination of four geometric distributions*. It is not a convex combination: (10) holds but, as we will see,

$$c_1 > 0, \quad c_2 < 0, \quad c_3 < 0, \quad c_4 < 0.$$

In particular, we have the inequality

$$t(n) < c_1 e_1^{n-1}, \qquad n \geq 1, \tag{11}$$

as well as the asymptotic formula

$$t(n) \sim c_1 e_1^{n-1} \quad \text{as } n \to \infty. \tag{12}$$

Another way to derive (9) is to begin with the recursive formula (3). The generating function of the tail probabilities (2) is

$$T(z) := \sum_{n=3}^{\infty} t(n) z^{n-1},$$

and by (3) we have

$$\begin{aligned} T(z) = &\left(1 - \sum_{j \in \mathscr{P}} \pi_j\right) z(z + T(z)) \\ &+ \sum_{j \in \mathscr{P}} \frac{\pi_j (1 - \pi_j - \pi_7) z^2}{1 - (1 - \pi_j - \pi_7) z} \\ &+ \sum_{j \in \mathscr{P}} \frac{\pi_j^2 z^2}{1 - (1 - \pi_j - \pi_7) z} (1 + z + T(z)). \end{aligned}$$

Solving for $T(z)$ using (1), we find that

$$T(z) = \frac{z^2(20736 - 33828z + 16346z^2 - 1975z^3)}{23328 - 58320z + 51534z^2 - 18321z^3 + 1975z^4},$$

the denominator of which can be written (cf. (8)) as

$$23328(1 - e_1 z)(1 - e_2 z)(1 - e_3 z)(1 - e_4 z).$$

A partial-fraction expansion leads to (9), except that $f$ is replaced by

$$f(w,x,y,z) := -\frac{1975 - 16346w + 33828w^2 - 20736w^3}{23328w^2(w-x)(w-y)(w-z)}.$$

Using Vieta's formulas, this alternative version of (9) can be shown to be equivalent to the original one; in fact, yet another version uses

$$f(w,x,y,z) := \frac{1975 - 16346w + 33828w^2 - 20736w^3}{3w^2(6107 - 34356w + 58320w^2 - 31104w^3)},$$

which has the advantage of depending only on $w$.

## Numerical Approximations

Rounding to 18 decimal places, the non-unit eigenvalues of $\boldsymbol{P}$ are

$$\begin{aligned} e_1 &\approx 0.862\,473\,751\,659\,322\,030, \\ e_2 &\approx 0.741\,708\,271\,459\,795\,977, \\ e_3 &\approx 0.709\,206\,775\,794\,379\,015, \\ e_4 &\approx 0.186\,611\,201\,086\,502\,979, \end{aligned}$$

and the coefficients in (9) are

$$\begin{aligned} c_1 &\approx \phantom{-}1.211\,844\,812\,464\,518\,572, \\ c_2 &\approx -0.006\,375\,542\,263\,784\,777, \\ c_3 &\approx -0.004\,042\,671\,248\,651\,503, \\ c_4 &\approx -0.201\,426\,598\,952\,082\,292. \end{aligned}$$

These numbers will give very accurate results over a wide range of values of $n$.

The result (12) shows that the leading term in (9) may be adequate for large $n$; it can be shown that

$$1 < c_1 e_1^{n-1} / t(n) < 1 + 10^{-m}$$

for $m = 3$ if $n \geq 19$; for $m = 6$ if $n \geq 59$; for $m = 9$ if $n \geq 104$; and for $m = 12$ if $n \geq 150$.

## Crapless Craps

In crapless craps [7, p. 354], as the name suggests, there are no craps numbers and 7 is the only natural. Therefore, the set of possible point numbers is

$$\mathscr{P}_0 := \{2, 3, 4, 5, 6, 8, 9, 10, 11, 12\}$$

but otherwise the rules of craps apply. More precisely, the pass-line bet is won either by rolling 7 on the come-out roll or by rolling a number other than 7 on the come-out roll and repeating that number before 7 appears.

With $L_0$ denoting the length of the shooter's hand, the analogues of (4)–(6) are

$$\begin{aligned} S_0 &:= \{\text{co, p2-12, p3-11, p4-10, p5-9, p6-8, 7o}\} \\ &\equiv \{1, 2, 3, 4, 5, 6, 7\}, \end{aligned}$$

$$\boldsymbol{P}_0 := \frac{1}{36} \begin{pmatrix} 6 & 2 & 4 & 6 & 8 & 10 & 0 \\ 1 & 29 & 0 & 0 & 0 & 0 & 6 \\ 2 & 0 & 28 & 0 & 0 & 0 & 6 \\ 3 & 0 & 0 & 27 & 0 & 0 & 6 \\ 4 & 0 & 0 & 0 & 26 & 0 & 6 \\ 5 & 0 & 0 & 0 & 0 & 25 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix},$$

and

$$t_0(n) := P(L_0 \geq n) = 1 - (\boldsymbol{P}_0^{n-1})_{1,7}.$$

There is an interesting distinction between this game and regular craps. The non-unit eigenvalues of $\boldsymbol{P}_0$ are the roots of the sextic equation

$$\begin{aligned} 0 = {}& 15116544z^6 - 59206464z^5 + 93137040z^4 \\ &- 73915740z^3 + 30008394z^2 - 5305446z + 172975, \end{aligned}$$

and the corresponding Galois group is, according to *Maple*, the symmetric group $S_6$. This means that our sextic is not solvable by radicals. Thus, it appears that there is no closed-form expression for $t_0(n)$.

Nevertheless, the analogue of (9) holds (with six terms). All non-unit eigenvalues belong to $(0, 1)$ and all coefficients except the leading one are negative. Thus, the analogues of (11) and (12) hold as well. Also, the distribution of $L_0$ is *a linear combination of six geometric distributions*. These results are left as exercises for the interested reader.

Finally, $t_0(154) \approx 0.296\,360\,068 \times 10^{-10}$, which is to say that a hand of length 154 or more is only about one-sixth as likely as at regular craps (one chance in 33.7 billion, approximately).


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