Take-home final exam, Math 6040

December 11, 2015

Directions: There are five problems. You may consult the text but may not discuss this exam with anyone else. The deadline is December 18 at 3 p.m. On the last page write "I did not give or receive assistance on this exam." and sign your name if it is true.

1. Prove that X_n converges to X in probability if and only if for every subsequence $\{n_k\}$ of the positive integers there exists a further subsequence $\{n_{k_i}\}$ such that $X_{n_{k_i}} \to X$ a.s.

2. Prove that if $X_1, X_2, \dots$ is an i.i.d. sequence in L^1, then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{j=1}^{n} \frac{X_j}{j} = EX_1 \quad \text{a.s.}$$

3. Consider the function $f(x) = (1 - |x|)_+$ on $\mathbf{R}$.
(a) If X has density f, find the characteristic function of X.
(b) Use the inversion theorem to show that f is the characteristic function of a probability distribution.

4. Let $X_1, X_2, \dots$ be an i.i.d. sequence of $N(0,1)$ random variables and $S_n = X_1 + \cdots + X_n$.
(a) Show that

$$M_n = \frac{1}{\sqrt{n+1}} \exp\{S_n^2/(2n+2)\}$$

is a mean 1 martingale.
(b) Use the martingale convergence theorem and the central limit theorem to show that $\lim_{n\to\infty} M_n = 0$ a.s.

5. Given $a \neq 0$ define $T_a = \inf\{s \geq 0 : B_s = a\}$, where B is standard Brownian motion.
(a) Show that the density of T_a is given by

$$f(x) = \frac{|a| e^{-a^2/(2x)}}{x^{3/2}\sqrt{2\pi}}, \quad x > 0.$$

(b) Show that the stochastic process $\{T_a,\, a \geq 0\}$ has i.i.d. increments.