Example 5.1.2. *Le her.* The game of le her ("the gentleman" in 17th-century French) is a two-person game played with a standard 52-card deck, and we will continue to refer to the two players as player 1 and player 2. Cards are ranked from lowest to highest in the order A, 2, 3, ..., 10, J, Q, K, and suits do not count. A card is dealt face down to each player, and each player may look only at his own card. The object of the game is to have the higher-ranking card at the end of play. First, player 1, if he is not satisfied with his card, can require that player 2 exchange cards with him. The only exception to this rule occurs when player 2 has a king (K), in which case the exchange is void. Second, player 2, if he is not satisfied with his card, whether it be his original card or a new card obtained in exchange with player 1, can exchange it for the next card in the deck. The only exception to this rule occurs when the next card is a king, in which case the exchange is void. This completes the game, and the winner is the player with the higher-ranked card, with player 2 winning in the case of a tie. The game pays even money.

It will be convenient to refer to cards A, 2, 3, ..., 10, J, Q, K as ranked 1, 2, 3, ..., 10, 11, 12, 13, respectively. Let us denote by X, Y, and Z the ranks of the card dealt to player 1, the card dealt to player 2, and the next card in the deck, respectively. Player 1's strategies correspond to the subsets $S \subset \{1, 2, \ldots, 13\}$. Given such an S, player 1 exchanges his card with that of player 2 if and only if $X \in S$. Player 2's strategies correspond to the subsets $T \subset \{1, 2, \ldots, 13\}$. Given such a T, if player 1 fails to exchange his card with that of player 2, player 2 exchanges his card with the next card in the deck if and only if $Y \in T$. Of course, if player 1 exchanges his card with that of player 2, then player 2's strategy is clear: He keeps his new card if $X \geq Y$ and exchanges it for the next card in the deck otherwise. Thus, we have a $2^{13} \times 2^{13}$ matrix game.

It is intuitively clear that the only reasonable strategies are of the form $S_i := \{1, \ldots, i\}$ and $T_j := \{1, \ldots, j\}$ for $i, j = 0, 1, \ldots, 13$ (of course, $S_0 = T_0 := \varnothing$). It can be shown that every other strategy is strictly dominated by at least one of these. (See Problem 5.7.)

Let B_{ij} denote the event that player 1 wins when player 1 uses strategy S_i and player 2 uses strategy T_j, $i, j = 0, 1, \ldots, 13$. We evaluate $\mathrm{P}(B_{ij})$ by conditioning on $\{X = k,\ Y = l\}$. There are three cases to consider.

Case 1. $k \leq i$. Here player 1 exchanges his card with that of player 2, provided player 2 does not have a king. The only case in which player 1 can win is $k < l < 13$, which forces player 2 to exchange his new card with the next card in the deck. Player 1 wins if $Z < l$ or if $Z = 13$. Therefore,

$$\begin{aligned} \mathrm{P}(B_{ij} \mid X = k,\ Y = l) &= \mathrm{P}(Z < l \text{ or } Z = 13 \mid X = k,\ Y = l)\, 1_{\{k<l<13\}} \\ &= \left(\frac{4(l-1)-1}{50} + \frac{4}{50}\right) 1_{\{k<l<13\}} \\ &= \frac{4l-1}{50}\, 1_{\{k<l<13\}}. \end{aligned} \tag{5.7}$$

Case 2. $k > i,\ l \le j$. Here player 1 keeps his card, while player 2 exchanges his card with the next card in the deck. Player 1 wins if $Z < k$ or if $Z = 13$ and $k > l$. Therefore,

$$\begin{aligned}
\mathrm{P}(B_{ij} \mid X = k,\ Y = l) &= \mathrm{P}(Z < k \mid X = k,\ Y = l) \\
&\qquad + \mathrm{P}(Z = 13 \mid X = k,\ Y = l)\, 1_{\{k>l\}} \\
&= \frac{4(k-1) - 1_{\{k>l\}}}{50} + \frac{4 - \delta_{k,13}}{50}\, 1_{\{k>l\}} \\
&= \frac{4(k-1) + (3 - \delta_{k,13})\, 1_{\{k>l\}}}{50}.
\end{aligned} \tag{5.8}$$

Case 3. $k > i,\ l > j$. Here both players keep their cards, so

$$\mathrm{P}(B_{ij} \mid X = k,\ Y = l) = 1_{\{k>l\}}.$$

It follows that

$$\begin{aligned}
\mathrm{P}(B_{ij}) = \sum_{k=1}^{i} \sum_{l=1}^{13} \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4l-1}{50}\, 1_{\{k<l<13\}} \\
+ \sum_{k=i+1}^{13} \sum_{l=1}^{j} \frac{4}{52} \cdot \frac{4 - \delta_{k,l}}{51} \cdot \frac{4(k-1) + (3 - \delta_{k,13})\, 1_{\{k>l\}}}{50} \\
+ \sum_{k=i+1}^{13} \sum_{l=j+1}^{13} \frac{4}{52} \cdot \frac{4}{51}\, 1_{\{k>l\}}.
\end{aligned} \tag{5.9}$$

Of course, this formula could be further simplified. For example, the first double sum could be written as a cubic polynomial in i. However, there is no need to do this, since our only concern is with the numerical evaluation of (5.9), and this is most reliably done by computer. The payoff matrix $\boldsymbol{A}$ for this game has (i, j) entry

$$a_{ij} = 2\mathrm{P}(B_{ij}) - 1. \tag{5.10}$$

The full matrix, multiplied by $(52)_3/2^3 = 16{,}575$, is displayed in Table 5.1.

The payoff matrix can be reduced considerably using strict dominance. Examining it, we see that strategies 0–4 for player 1 are strictly dominated by strategy 5 for player 1, and that strategies 8–13 for player 1 are strictly dominated by strategy 7 for player 1. Eliminating the strictly dominated strategies, we are left with the 3×14 payoff matrix displayed in Table 5.2.

Next, in Table (5.2) strategies 0–6 for player 2 are strictly dominated by strategy 7 for player 2, and strategies 9–13 for player 2 are strictly dominated by strategy 8 (or 7) for player 2. Eliminating the strictly dominated strategies, we are left with

$$\begin{array}{c} \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cc} 7 & 8 \end{array} \\ \begin{bmatrix} 105 & 221 \\ 393 & 429 \\ 453 & 393 \end{bmatrix} \end{array}. \tag{5.11}$$

Finally, in (5.11) strategy 5 for player 1 is strictly dominated by strategy 6 for player 1, so we end up with the 2×2 payoff matrix, multiplied by 16,575, of

$$\begin{array}{c} \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cc} 7 & 8 \end{array} \\ \begin{bmatrix} 393 & 429 \\ 453 & 393 \end{bmatrix} \end{array}. \tag{5.12}$$

In the next section we will find the optimal strategies for players 1 and 2. ♠

The next example is even more complicated, but it can be simplified considerably with the aid of the following lemma.

Table 5.1 Payoff matrix, multiplied by 16,575, for the game of le her.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	−975	−1987	−2815	−3459	−3919	−4195	−4287	−4195	−3919	−3459	−2815	−1987	−975	225
1	213	−799	−1627	−2271	−2731	−3007	−3099	−3007	−2731	−2271	−1627	−799	213	1413
2	1173	333	−507	−1167	−1643	−1935	−2043	−1967	−1707	−1263	−635	177	1173	2357
3	1889	1205	521	−163	−671	−995	−1135	−1091	−863	−451	145	925	1889	3041
4	2345	1801	1257	713	169	−203	−391	−395	−215	149	697	1429	2345	3449
5	2525	2105	1685	1265	845	425	173	105	221	521	1005	1673	2525	3565
6	2413	2101	1789	1477	1165	853	541	393	429	649	1053	1641	2413	3373
7	1993	1773	1553	1333	1113	893	673	453	393	517	825	1317	1993	2857
8	1249	1105	961	817	673	529	385	241	97	109	305	685	1249	2001
9	165	81	−3	−87	−171	−255	−339	−423	−507	−591	−523	−271	165	789
10	−1275	−1315	−1355	−1395	−1435	−1475	−1515	−1555	−1595	−1635	−1675	−1567	−1275	−795
11	−3087	−3099	−3111	−3123	−3135	−3147	−3159	−3171	−3183	−3195	−3207	−3219	−3087	−2767
12	−5287	−5287	−5287	−5287	−5287	−5287	−5287	−5287	−5287	−5287	−5287	−5287	−5287	−5143
13	−7687	−7687	−7687	−7687	−7687	−7687	−7687	−7687	−7687	−7687	−7687	−7687	−7687	−7687

Table 5.2 Payoff matrix, multiplied by 16,575, for the game of le her, after eliminating 11 strictly dominated rows.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
5	2525	2105	1685	1265	845	425	173	105	221	521	1005	1673	2525	3565
6	2413	2101	1789	1477	1165	853	541	393	429	649	1053	1641	2413	3373
7	1993	1773	1553	1333	1113	893	673	453	393	517	825	1317	1993	2857