

You have a choice of any four of the five problems. (If you do all 5, each will count $1/5$, meaning there is no advantage.) This is a closed-book exam, and calculators are not allowed or needed. Cell phone/Internet use is prohibited. Show your work so that you can get partial credit in the case of a wrong answer.

1. 1. A horizontal line of n coins is laid out randomly with some coins showing heads and some tails. A move consists of turning over one of the coins from heads to tails, and in addition, if desired, turning over one other coin anywhere to the left of it (from heads to tails or tails to heads). For example consider the sequence of $n = 13$ coins:

T	H	T	T	H	T	T	T	H	H	T	H	T
1	2	3	4	5	6	7	8	9	10	11	12	13

One possible move in this position is to turn the coin in place 9 from heads to tails, and also the coin in place 4 from tails to heads. The last player to make a legal move wins.

(a) Carefully explain why this game is just nim in disguise if an H in place n is taken to represent a nim pile of n chips. (In particular, you'll need to explain what corresponds to the move above in which you turn over the coin in place 9 from heads to tails and the coin in place 2 from heads to tails.)

Sol. There are three types of moves. (a) Turn over H to T at n . This corresponding to removing all chips in the pile of size n . (b) Turning over H to T at n and T to H at $k < n$. This corresponds to removing $n - k$ chips from the pile of size n , which becomes a pile of size k . (c) Turning over H to T at n and H to T at $k < n$. This corresponds to removing $n - k$ chips from the pile of size n . The point is that eliminating piles of size n and size k is essentially the same as removing $n - k$ chips from the pile of size n , leaving two piles of size k . Two piles of size k is equivalent to no piles because $k \oplus k = 0$.

(b) Assuming (a) to be true, find all winning first moves from the above position.

Sol. Since $2 \oplus 5 \oplus 9 \oplus 10 \oplus 12 = 8$, we must reduce one of the piles of size 9, 10, or 12 by 8. We can do this by turning over the coins in places 12 and 4, 10 and 2, or 9 and 1.

2. (a) Compute the Sprague–Grundy function for Grundy’s game (you may split a single pile into two nonempty piles of different sizes) of n chips for $n = 1, 2, \dots, 10$.

Sol.

x	$F(x)$	$g(F(x))$	$g(x)$
1	\emptyset	\emptyset	0
2	\emptyset	\emptyset	0
3	(1, 2)	0	1
4	(1, 3)	1	0
5	(1, 4), (2, 3)	0, 1	2
6	(1, 5), (2, 4)	0, 2	1
7	(1, 6), (2, 5), (3, 4)	1, 2	0
8	(1, 7), (2, 6), (3, 5)	0, 1, 3	2
9	(1, 8), (2, 7), (3, 6), (4, 5)	0, 2	1
10	(1, 9), (2, 8), (3, 7), (4, 6)	1, 2	0

(b) In Grundy’s game with three piles of sizes 5, 7, and 10, find all winning first moves, if any.

Sol. $g(5) \oplus g(7) \oplus g(10) = 2 \oplus 0 \oplus 0 = 2$, so we must split pile 1 in such a way that the SG-value of the split piles is 0. The only way to split a pile of 5 is (1, 4) and (2, 3). The first has SG value 0 and the second has SG value 1. So there is a unique winning first move, namely split the pile of 5 into piles of sizes 1 and 4.

Correction: There are two other winning moves. Since $2 \oplus 2 \oplus 0 = 0$ and $2 \oplus 0 \oplus 2 = 0$, winning moves include splitting the pile of 7 into piles of 2 and 5; splitting the pile of 10 to piles of 2 and 8.

3. A 2×2 game has payoff matrix of the form

$$\begin{pmatrix} a & b \\ d & c \end{pmatrix}.$$

(a) Give an example with specific values for a, b, c, d such that $(1, 2)$ is a saddle point, hence row 1 is optimal for I, column 2 is optimal for II, and the value of the game is b .

Sol. We want b to be a row min and a column max. So $a \geq b$ and $b \geq c$. Examples include

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}.$$

(b) Give an example with specific values for a, b, c, d such that the game has value 0 but the optimal mixed strategies for I and II are different.

Sol. Let's do this without saddle points, so either $\min(a, c) > \max(b, d)$ or $\min(b, d) > \max(a, c)$. Then our formula applies, so we need $ac - bd = 0$ and $c - d \neq c - b$ or $b \neq d$. Examples include

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} -6 & 9 \\ 4 & -6 \end{pmatrix}.$$

We could also do this with either $(1, 2)$ or $(2, 1)$ being a saddle point. For $(1, 2)$ to be a saddle point, we have already seen that we need $a \geq b$ and $b \geq c$, and for this part we'd also need $b = 0$. For $(2, 1)$ to be a saddle point, we would need d to be a row min and column max, or $c \geq d$ and $d \geq a$, and for this part we'd also need $d = 0$.

(c) Give an example with specific values for a, b, c, d such that the game has a nonzero value but the optimal mixed strategies for I and II are identical.

Sol. Again, let's do this without saddle points, so either $\min(a, c) > \max(b, d)$ or $\min(b, d) > \max(a, c)$. Then our formula applies, so we need $ac - bd \neq 0$ and $c - d = c - b$ or $b = d$. Examples include

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}.$$

We could also do this with either $(1, 1)$ or $(2, 2)$ being a saddle point. For $(1, 1)$ to be a saddle point, we would need a to be a row min and a column max, or $b \geq a$ and $a \geq d$, and for this part we'd also need $a \neq 0$. For $(2, 2)$ to be a saddle point, we would need c to be a row min and column max, or $d \geq c$ and $c \geq b$, and for this part we'd also need $c \neq 0$.

4. A magic square is an $n \times n$ array of the first n^2 integers with the property that all row and column sums are equal. Solve the game with 4×4 magic square payoff matrix

$$\begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}.$$

Specifically, guess optimal mixed strategies for players I and II and the value of the game. Then verify that your guesses are correct.

Sol. Call the matrix \mathbf{A} . Because row and column sums are equal to 34, we may suspect that the uniform distribution is optimal for both players. To check this, let $\mathbf{p} = \mathbf{q} = (1/4, 1/4, 1/4, 1/4)^T$. Then $\mathbf{p}^T \mathbf{A} = (1/4)(34, 34, 34, 34)$ and $\mathbf{A} \mathbf{q} = (1/4)(34, 34, 34, 34)^T$. We conclude that with $V = 34/4 = 8.5$, $\mathbf{p}^T \mathbf{A} \geq V \mathbf{1}^T$ and $\mathbf{A} \mathbf{q} \leq V \mathbf{1}$ (actually, equality holds in both cases), so \mathbf{p} and \mathbf{q} are optimal strategies and V is the value of the game.

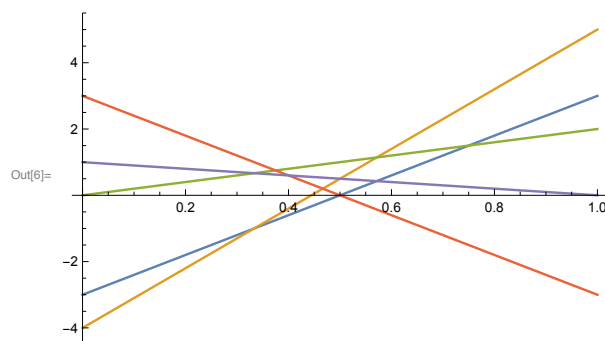
5. Solve the game with 5×2 payoff matrix

$$\begin{pmatrix} 3 & -3 \\ 5 & -4 \\ 2 & 0 \\ -3 & 3 \\ 0 & 1 \end{pmatrix}.$$

In other words, find optimal mixed strategies for players I and II, and find the value of the game.

Sol. Here we cheat slightly and use a computer to plot the straight lines.

In[6]:= `Plot[{3 q - 3 (1 - q), 5 q - 4 (1 - q), 2 q, -3 q + 3 (1 - q), 1 - q}, {q, 0, 1}]`



We see that the upper envelope is minimized at about $q = 0.375$ at the intersection of the red line connecting $(0, 3)$ and $(1, -3)$ and the green line connecting $(0, 0)$ and $(1, 2)$. So it suffices to solve the game

$$\begin{pmatrix} 2 & 0 \\ -3 & 3 \end{pmatrix}$$

corresponding to rows 3 and 4 of the 5×2 game. There is no saddle point, so the optimal strategy for I is $(3/4, 1/4)^T$, the optimal strategy for II is $(3/8, 5/8)^T$, and the game's value is $3/4$. Returning to the original 5×2 game, the optimal strategy for I is $\mathbf{p} = (0, 0, 3/4, 1/4, 0)^T$, the optimal strategy for II is $\mathbf{q} = (3/8, 5/8)^T$, and the game's value is $V = 3/4$.