Math 5750-1: Game Theory
Midterm Exam with solutions
Mar. 6, 2015

You have a choice of any four of the five problems. (If you do all 5, each will count 1/5, meaning there is no advantage.) This is a closed-book exam, and calculators are not allowed or needed. Cell phone/Internet use is prohibited. Show your work so that you can get partial credit in the case of a wrong answer.

1. A position in the game of Rims is a finite set of dots in the plane, possibly separated by some nonintersecting closed loops. A move consists of drawing a closed loop passing through any positive number of dots (at least one) but not touching any other loop. Players alternate moves and the last to move wins.

(a) Explain why this game is a disguised form of nim.

Sol. Instead of several piles of chips, we have several clusters of dots separated by loops (excluding those that have been crossed by a loop). Instead of removing k chips from a pile, we can draw a closed loop through k of the dots in a cluster. (Technically, there is a small difference between this game and nim. You can split a pile after you remove a chip by drawing your loop to enclose some but not all of the dots.)

(b) In the position given in the figure below, find a winning move, if any.

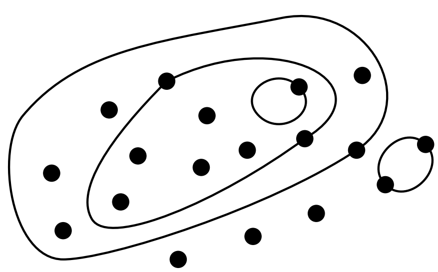

Figure 1: A position in the game of Rims.

Sol. The cluster sizes are 4, 5, and 3. So we play nim with pile sizes 4, 5, and 3.

$$4 = 1\ 0\ 0$$
$$5 = 1\ 0\ 1$$
$$3 = 0\ 1\ 1$$

The nim sum is $0\ 1\ 0 = 2$, so we must reduce pile 3 to 1 chip to make the nim sum equal to 0. The equivalent move in Rims is to draw a loop through two of the three dots that are outside of each of the loops.

2. Suppose that at each turn a player may select one pile and remove c chips if $c-1$ is divisible by 3 and, if desired, split the remaining chips into two piles.

(a) Find the Sprague–Grundy function $g(x)$ for a pile of size $x = 0, 1, \ldots, 8$. (Check your work—it's easy to make a mistake.)

Sol. We use a table to find the SG function $g(x)$. Some students read the rules carelessly. Notice that we must remove 1, 4, or 7 chips (if pile size is 8 or less) and then we may or may not split the remainder of that pile into two piles.

x	$F(x)$	$g(F(x))$	$g(x)$
0	$\varnothing$	$\varnothing$	0
1	0	0	1
2	1	1	0
3	$2, (1,1)$	0	1
4	$0, 3, (1,2)$	$0, 1$	2
5	$1, 4, (1,3), (2,2)$	$0, 1, 2$	3
6	$2, 5, (1,1), (1,4), (2,3)$	$0, 1, 3$	2
7	$0, 3, 6, (1,2), (1,5), (2,4), (3,3)$	$0, 1, 2$	3
8	$1, 4, 7, (1,3), (2,2), (1,6), (2,5), (3,4)$	$0, 1, 2, 3$	4

(b) Find a winning first move if initially there are piles of sizes 6, 7, and 8.

Sol. $g(6) \oplus g(7) \oplus g(8) = 2 \oplus 3 \oplus 4 = 5$ To make this 0, we can change it to $2 \oplus 3 \oplus 1 = 0$, so we must reduce the pile of 8 to a size that has g-value 1. That means reducing it to 3 chips or 1 chip. Only one of these is a legal move, so we must reduce the pile of 8 to a single chip.

3. (a) Find a 2×2 payoff matrix $\boldsymbol{A}$ with optimal strategies $\boldsymbol{p}^* = (4/7, 3/7)^T$ for player I and $\boldsymbol{q}^* = (5/7, 2/7)^T$ for player II.

Sol. We apply the formulas for the optimal $\boldsymbol{p}^* = (p^*, 1 - p^*)^T$ and $\boldsymbol{q}^* = (q^*, 1 - q^*)^T$.

$$p^* = \frac{c - d}{a - b + c - d} = \frac{4}{7} \quad \text{if} \quad a - b = 3 \text{ and } c - d = 4.$$

$$q^* = \frac{c - b}{a - b + c - d} = \frac{5}{7} \quad \text{if} \quad a - d = 2 \text{ and } c - b = 5.$$

It follows that $b = a - 3$, $c = b + 5 = a + 2$, and $d = c - 4 = a - 2$, so the desired matrix is

$$\begin{pmatrix} a & a - 3 \\ a - 2 & a + 2 \end{pmatrix}$$

for an arbitrary a. Take $a = 0$ to get one possible answer,

$$\begin{pmatrix} 0 & -3 \\ -2 & 2 \end{pmatrix},$$

which has value $V = -6/7$.

(b) By adding a constant to each entry of $\boldsymbol{A}$ if necessary, arrange it so that the value of the game is $V = 1/7$. (The optimal strategies will not change.)

Sol. We can add 1 to each entry (or take $a = 1$ above) to get

$$\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix},$$

which has value $V = -6/7 + 1 = 1/7$.

4. In Mendelsohn games, two players simultaneously choose a positive integer. Both players want to choose an integer larger but not too much larger than the opponent. Here is a simple example. The players choose an integer between 1 and 100. If the numbers are equal there is no payoff. The player that chooses a number one larger than that chosen by his opponent wins 1. The player that chooses a number two or more larger than his opponent loses 2. The payoff matrix is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \end{array} \\ \begin{pmatrix} 0 & -1 & 2 & 2 & 2 & \cdots \\ 1 & 0 & -1 & 2 & 2 & \cdots \\ -2 & 1 & 0 & -1 & 2 & \cdots \\ -2 & -2 & 1 & 0 & -1 & \cdots \\ -2 & -2 & -2 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}.$$

(a) Eliminate dominated strategies, reducing the game to a 3×3 game.

Sol. We notice that row 1 dominates rows 4, 5, 6, and so on. By symmetry, column 1 dominates columns 4, 5, 6, and so on. We are left with rows 1–3 and columns 1–3, that is,

$$\boldsymbol{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \end{array}$$

(b) Solve the 3×3 game by finding an optimal mixed strategy for player I. You may guess a mixed strategy, use a formula, or use the equilibrium theorem. In any case, verify that your mixed strategy for player I is indeed optimal. (The game is symmetric, so the value of the game is 0 and an optimal mixed strategy for player I is also optimal for player II.)

Sol. We had a formula for the solution of a 3×3 symmetric game, which implies that $(\boldsymbol{p}^*)^T = (1/4, 1/2, 1/4)$. If you didn't remember the formula, you could have used the equilibrium theorem and solve the system $\boldsymbol{Ap} = \boldsymbol{0}$ with $p_1 + p_2 + p_3 = 1$. This gives $-p_2 + 2p_3 = 0$, $p_1 - p_3 = 0$, $-2p_1 + p_2 = 0$, and $p_1 + p_2 + p_3 = 0$. Thus, $p_1 = p_3$ and p_2 is twice p_1. In other words, p_1, p_2, p_3 are proportional to $1, 2, 1$, and the stated result follows.

To verify that this $\boldsymbol{p}^*$ is a solution, it is enough to show that $(\boldsymbol{p}^*)^T \boldsymbol{A} = \boldsymbol{0}^T$, or $\boldsymbol{Ap}^* = \boldsymbol{0}$, both of which are immediate.

Several people came up with $(\boldsymbol{p}^*)^T = (1/3, 1/3, 1/3)$, but they neglected to check whether this is a solution. Note that $\boldsymbol{Ap}^* = (1/3, 0, -1/3)^T$.

5. Solve the game with payoff matrix

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 11 & -5 & 7 & 1 & -2 \\ -10 & 6 & -8 & -6 & 4 \end{pmatrix}, \end{array}$$

i.e., find the value of the game and optimal strategies for player I (row player) and player II (column player) in terms of the original game.

Sol. Here we cheat slightly and use a computer to plot the straight lines.

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Plot[{11 p - 10 (1 - p), -5 p + 6 (1 - p),
  7 p - 8 (1 - p), p - 6 (1 - p), -2 p + 4 (1 - p)}, {p, 0, 1}]
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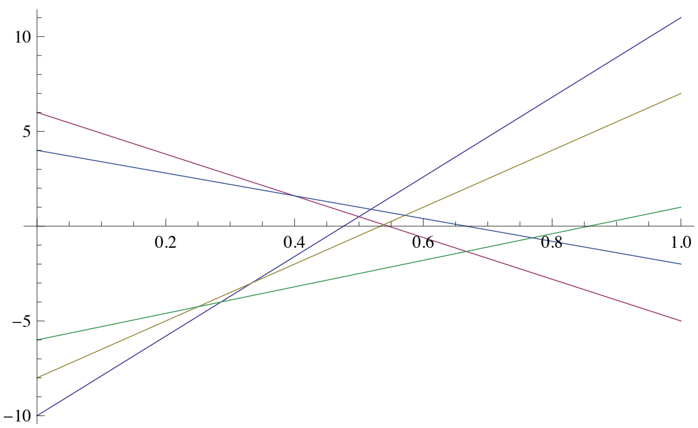

We see that the lower envelope is maximized at about $p = 2/3$ at the intersection of the line connecting $(0, -6)$ and $(1, 1)$ and the line connecting $(0, 6)$ and $(1, -5)$. So it suffices to solve the game

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 2 & 4 \end{array} \\ \begin{pmatrix} -5 & 1 \\ 6 & -6 \end{pmatrix}. \end{array}$$

There is no saddle point, so the optimal strategy for player I is $(2/3, 1/3)^T$ and the optimal strategy for player II is $(7/18, 11/18)^T$, and the game's value is $V = -4/3$. Returning to the original 2×5 game, the optimal strategy for player I is $\boldsymbol{p} = (2/3, 1/3)^T$ and for II is $\boldsymbol{q} = (0, 7/18, 0, 11/18, 0)^T$, and the game's value is $V = -4/3$.