5750 2016 Final

① remove 1 if whole pile
remove 2 or more and, if desired split.

0	$\emptyset$	$\emptyset$	0
1	0	0	1
2	0	0	1
3	0, 1	0, 1	2
4	0, 1, 2, (1,1)	0, 1	2
5	0, 1, 2, 3, (1,1), (1,2)	0, 1, 2	3
6	0, 1, 2, 3, 4, (1,1), (1,2), (1,3), (2,2)	0, 1, 2, 3	4
7	0–5, (1,1), (1,2), (1,3), (2,2), (1,4), (2,3)	0–3	4
8	0–6, (1,1), (1,2), (1,3), (2,2), (1,4), (2,3), (1,5), (2,4), (3,3)	0–4	5
9	0–7, all of above, (1,6), (2,5), (3,4)	0–4, 5	6

$g(5,6,8) = 3 \oplus 4 \oplus 5$ 3 → 1

$$\begin{array}{r} 011 \\ 100 \\ 101 \\ \hline 010 \end{array} \quad 011 \to 001$$

follower of 5 with g value 1

Reduce pile of 5 to 1 or 2.
These are the only winning moves.

2

$$(p_1, p_2, p_3) = V\,(1,1,1)A^{-1} = V\,\frac{1}{64}(10, 9, 8)$$

so $\boxed{V = \frac{64}{27}}$ $\boxed{= \left(\frac{10}{27}, \frac{9}{27}, \frac{8}{27}\right)}$

$$A\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} V \\ V \\ V \end{pmatrix} \Rightarrow \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} V$$

$$= \frac{1}{64}\begin{pmatrix} 24 \\ 2 \\ 1 \end{pmatrix} V = \boxed{\begin{pmatrix} \frac{24}{27} \\ \frac{2}{27} \\ \frac{1}{27} \end{pmatrix}}$$

Alt., $V = \left(\underset{\sim}{1}\,\underset{\sim}{A}^{-1}\,1\right)^{-1} = \left(\frac{1}{64}\cdot 27\right)^{-1} = \frac{64}{27}$.

$$p = \frac{1^T A^{-1}}{1^T A^{-1} 1} = \frac{(10, 9, 8)}{27}$$

$$q = \frac{A^{-1} 1}{1^T A^{-1} 1} = \begin{pmatrix} 24 \\ 2 \\ 1 \end{pmatrix}\frac{1}{27}$$

(3) I: $\{a, b, c\}$

II: $\{d, e\} \times \{f, g\}$

The payoff matrix is 3×4

$$
\begin{array}{c c}
 & \begin{array}{cccc} df & dg & ef & eg \end{array} \\
\begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} \frac{3+2(-3)}{3} = -1 & \frac{3+2(-3)}{3} = -1 & \frac{3+2(0)}{3} = 1 & \frac{3+2\cdot 3}{3} = 3 \\ \frac{3+2(0)}{3} = 1 & \frac{0+2(0)}{3} = 0 & \frac{3+2(0)}{3} = 1 & \frac{0+2(3)}{3} = 2 \\ \frac{3+2(0)}{3} = 1 & \frac{3+2(0)}{3} = 1 & \frac{-3+2(0)}{3} = -1 & \frac{-3+2(3)}{3} = 1 \end{pmatrix}
\end{array}
$$

dom dom dom

$$
\begin{pmatrix} -1 & -1 & 1 & 3 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & 1 \end{pmatrix}
$$

no saddle points

$$
\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}
$$

$\boxed{V = \frac{-1}{-3} = \frac{1}{3}}$ $\quad p = \left(\frac{2}{3}, \frac{1}{3}\right) \quad q = \left(\frac{2}{3}, \frac{1}{3}\right)$

$\boxed{p = \left(0, \frac{2}{3}, \frac{1}{3}\right)}$, $\boxed{q = \left(0, \frac{2}{3}, \frac{1}{3}, 0\right)}$

4

$$\begin{pmatrix} (a_1^*, b_1^*) & (0,0) & (0,0) \\ (0,0) & (a_2^*, b_2^*) & (0,0) \\ (0,0) & (0,0) & (a_3^*, b_3^*) \end{pmatrix}$$

3 pure NES. $\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle$.

For a mixed NE, find equalizing strategy for opponent's matrix.

$$(p_1, p_2, p_3)\begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix} = (V, V, V)$$

$$(p_1 b_1, p_2 b_2, p_3 b_3) = (V, V, V)$$

$$(p_1, p_2, p_3) = V(b_1^{-1}, b_2^{-1}, b_3^{-1}) = \boxed{\frac{(b_1^{-1}, b_2^{-1}, b_3^{-1})}{b_1^{-1} + b_2^{-1} + b_3^{-1}}}$$

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} V' \\ V' \\ V' \end{pmatrix}$$

$$a_1 q_1 = V' = a_2 q_2 = a_3 q_3$$

$$(q_1, q_2, q_3) = \boxed{\frac{(a_1^{-1}, a_2^{-1}, a_3^{-1})}{a_1^{-1} + a_2^{-1} + a_3^{-1}}}$$

This pair is an NE.

(5) $(A,B) = \begin{pmatrix} (1,0) & (-1,1) & (0,0) \\ (3,3) & (-2,9) & (2,7) \end{pmatrix}$

$A = \begin{pmatrix} 1 & \boxed{-1} & 0 \\ 3 & -2 & 2 \end{pmatrix} \quad -B = \begin{pmatrix} 0 & \boxed{-1} & 0 \\ -3 & -9 & -7 \end{pmatrix}$ saddle points

$$\lambda A + B = \begin{pmatrix} \lambda & -\lambda+1 & 0 \\ 3\lambda+3 & -2\lambda+9 & 2\lambda+7 \end{pmatrix}$$

max entry in 3rd row

$-2\lambda+9 = 2\lambda+7$ $3\lambda+3 = 2\lambda+7$

$4\lambda = 2$ $\lambda = 4$

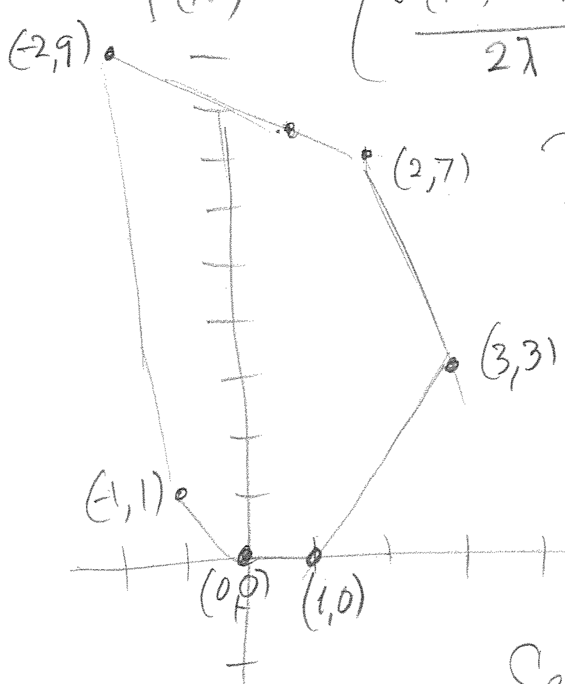
$\lambda = \frac{1}{2}$

max. is
$$\sigma(\lambda) = \begin{cases} -2\lambda+9 & \text{for } 0 < \lambda \le \frac{1}{2} \\ 2\lambda+7 & \text{for } \frac{1}{2} \le \lambda \le 4 \\ 3\lambda+3 & \text{for } \lambda \ge 4 \end{cases}$$
$\sigma(\frac{1}{2}) = 8$

$\sigma(4) = 15$

Value is $Val(\lambda A - B) = -\lambda - 1$

$$\varphi(\lambda) = \left(\frac{\sigma(\lambda) - \lambda - 1}{2\lambda}, \frac{\sigma(\lambda) + \lambda + 1}{2} \right)$$



By hint, $\sigma(\lambda) = 2\lambda + 7$ for $\frac{1}{2} < \lambda < 4$

So $\varphi(\lambda) = \left(\frac{\lambda+6}{2\lambda}, \frac{3\lambda+8}{2} \right)$

$= (2,7)$ if $\lambda = 2$

so $(\bar{u}, \bar{v}) = (2,7)$

and $\lambda^* = 2$

See Solution of Ex. 5(b), p. III-40 for alternative solution.

6 Core:

$$\left.\begin{array}{ll} x_1 \ge 0 & x_1 + x_3 \ge 1 \\ x_2 \ge 0 & x_2 + x_3 \ge 1 \\ x_3 \ge 0 & \\ x_1 + x_2 + x_3 = 1 & \end{array}\right\} \quad \left.\begin{array}{ll} x_1 + x_3 = 1 & x_2 = 0 \\ x_2 + x_3 = 1 & x_1 = 0 \end{array}\right\} \Rightarrow x_3 = 1$$

so $\{(0,0,1)\}$ is core (a single point)

Shapley:

$$\phi_1(v) = \frac{(2-1)!(3-2)!}{3!}\left[v(1,3) - v(3)\right] = \frac{1}{6}$$

$$\phi_2(v) = \frac{(2-1)!(3-2)!}{3!}\left[v(2,3) - v(3)\right] = \frac{1}{6}$$

$$\phi_3(v) = \frac{(2-1)!(3-2)!}{3!}\left[v(1,3) - v(1)\right] + \frac{(2-1)!(3-2)!}{3!}\left[v(2,3) - v(2)\right] + \frac{(3-1)!(3-3)!}{3!}\left[v(1,2,3) - v(1,2)\right]$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = \frac{2}{3}.$$

$$\boxed{\phi(v) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)}$$