Math 5750 Final – Spring 2015

(1)

(a)

10 pts.

3 omitted

x	legal subtractions	$F(x)$	$g(F(x))$	$g(x)$
0	$\emptyset$	$\emptyset$	$\emptyset$	0
1	1	0	0	1
2	1	1	1	0
3	1	2	0	1
4	1, 4	0, 3	0, 1	2
5	1, 4	1, 4	1, 2	0
6	1, 4	2, 5	0	1
7	1, 4	3, 6	1	0
8	1, 4	4, 7	0, 2	1
9	1, 4, 9	0, 5, 8	0, 1	2
10	1, 4, 9	1, 6, 9	1, 2	0
11	1, 4, 9	2, 7, 10	0	1
12	1, 4, 9	3, 8, 11	1	0

9

(b) $g(8) \oplus g(9) \oplus g(10) = 1 \oplus 2 \oplus 0 = 3$

10 pts.

$$\begin{array}{r} 01 \\ 10 \\ 00 \\ \hline 11 \end{array}$$

must change 01 to 10 *

or 10 to 01 **

or 00 to 11 not possible

* $8 \to 4$ works since $g(4) = 2$

** $9 \to 8$ works since $g(8) = 1$

There are only two winning moves.

Just one: -4 Use binary 8, 9, 10 : -6

(2)

II

I	1	2	3	4	5	6	7	8	9	10	11	12
1	0	−1	2	2	2	2	2	2	2	2	2	2
2	1	0	−1	−1	−1	2	2	2	2	2	2	2
3	−2	1	0	−1	−1	−1	−1	−1	2	2	2	2
4	−2	1	1	0	−1	−1	−1	−1	−1	−1	−1	2
5	−2	1	1	1	0	−1	−1	−1	−1	−1	−1	−1
6	−2	−2	1	1	1	0	−1	−1	−1	−1	−1	−1
7	−2	−2	1	1	1	1	0	−1	−1	−1	−1	−1
8	−2	−2	1	1	1	1	1	0	−1	−1	−1	−1
9	−2	−2	−2	1	1	1	1	1	0	−1	−1	−1
10	−2	−2	−2	1	1	1	1	1	1	0	−1	−1
11	−2	−2	−2	1	1	1	1	1	1	1	0	−1
12	−2	−2	−2	−2	1	1	1	1	1	1	1	0

3 omitted

(a) Row 1 str. doms rows 6–12
Col 1 str. doms cols 6–12

This leaves:

1	0	−1	2	2	2
2	1	0	−1	−1	−1
3	−2	1	0	−1	−1
4	−2	1	1	0	−1
5	−2	1	1	1	0

Row 5 doms. rows 3, 4
Col 5 doms cols 3, 4

	1	2	5
1	0	−1	2
2	1	0	−1
5	−2	1	0

(a) reduce to rows 1, 2, 5; cols 1, 2, 5

12 pts. 6 pts for matrix. 6 pts for dom.

(b) Solve 3×3

8 pts.

$$p_2 - 2p_5 = 0$$
$$-p_1 + p_5 = 0$$
$$2p_1 - p_2 = 0$$

$$(p_1, p_2, p_5) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$$

(3) game is 4 × 4 7 omitted

rows : 2
cols : 2
entries 6
(2 for mixture)

10 pts

$$\begin{array}{c|cccc} & ac & ad & bc & bd \\ AC & \frac{0+2}{2} & \frac{0+2}{2} & \frac{2-2}{2} & \frac{2-2}{2} \\ AD & \frac{0+0}{2} & \frac{0+4}{2} & \frac{2+0}{2} & \frac{2+4}{2} \\ BC & \frac{1+2}{2} & \frac{-1+2}{2} & \frac{1-2}{2} & \frac{-1-2}{2} \\ BD & \frac{1+0}{2} & \frac{-1+4}{2} & \frac{1+0}{2} & \frac{-1+4}{2} \end{array}$$

dom by $\frac{1}{2}$ row 3 + $\frac{1}{2}$ row 4

str dom by col 3

2+2

$$= \begin{array}{cccc|c} & & & & \text{row min} \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 1.5 & 0.5 & -0.5 & -1.5 & -1.5 \\ 0.5 & 1.5 & 0.5 & 1.5 & 0.5 \\ \hline \text{col max } 1.5 & 2 & 1 & 3 & \end{array}$$

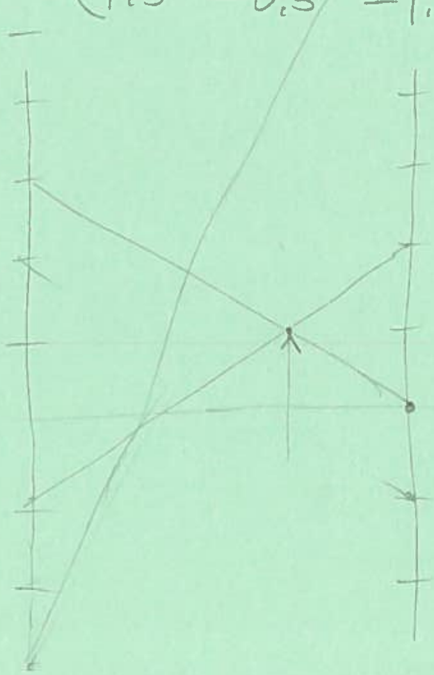
$$\begin{pmatrix} 0 & 1 & 3 \\ 1.5 & -0.5 & -1.5 \\ 0.5 & 0.5 & 1.5 \end{pmatrix}$$

row 3 dom by $\frac{2}{3}$ row 1 + $\frac{1}{3}$ row 2

2

$$\begin{pmatrix} 0 & 1 & 3 \\ 1.5 & -0.5 & -1.5 \end{pmatrix}$$

4



$$(1-p)\frac{3}{2} = p - (1-p)\frac{1}{2}$$

$$2-2p = p$$

$$p = \frac{2}{3} \quad V = \frac{1}{2}$$

AD w/p $\frac{2}{3}$

BC w/p $\frac{1}{3}$

$V = \frac{1}{2}$.

$$A = \begin{pmatrix} 0 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

ac w/p $\frac{1}{2}$

bc w/p $\frac{1}{2}$

④ $A = \begin{pmatrix} 3 & 2 & 3 \\ 6 & 0 & 3 \\ 4 & 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 4 & 5 \end{pmatrix}$ ~~none~~ 1 omitted

(a) $v_I = Val(A) = 3$ $\langle 3,2 \rangle$ is saddle pt. 3

max min $(0,0,1)$ 2

$$v_{II} = Val(B^T) = Val\begin{pmatrix} 4 & 1 & 6 \\ 3 & 2 & 4 \\ 2 & 3 & 5 \end{pmatrix}$$

$$= Val\begin{pmatrix} 4 & 1 \\ 3 & 2 \\ 2 & 3 \end{pmatrix} = Val\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = \frac{12-2}{3+1} = \frac{5}{2}$$ 3

max min $= \left(\frac{1}{4}, 0, \frac{3}{4}\right)$ 2

(b) Pure first:

$$(A,B) = \begin{pmatrix} (3,4^*) & (2,3) & (3,2) \\ (6^*,1) & (0,2) & (3,3^*) \\ (4,6^*) & (3^*,4) & (4^*,5) \end{pmatrix}$$ none 4

row 1 of A is strictly dom so $p_1 = 0$

$$\begin{pmatrix} (6,1) & (0,2) & (3,3) \\ (4,6) & (3,4) & (4,5) \end{pmatrix}$$

col 2 of B is strictly dom so

$$\begin{pmatrix} (6,1) & (3,3) \\ (4,6) & (4,5) \end{pmatrix}$$ 2

4

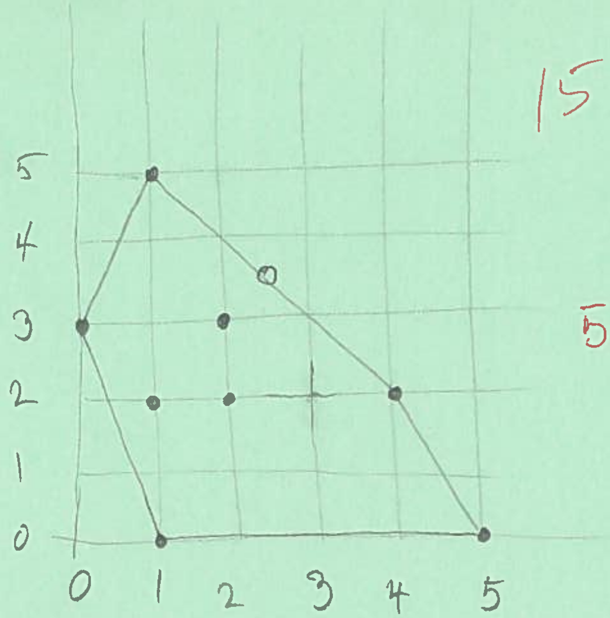
$(p_2, p_3)\begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}$ has equal entries $p_2 + 6p_3 = 3p_2 + 5p_3$

or $2p_2 - p_3 = 0$ $p_2 + p_3 = 1$ $p = \left(0, \frac{1}{3}, \frac{2}{3}\right)$

$\begin{pmatrix} 6 & 3 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} q_1 \\ q_3 \end{pmatrix}$ has eq. entries $6q_1 + 3q_3 = 4q_1 + 4q_3$

$2q_1 - q_3 = 0$ $q_1 + q_3 = 1$ $q = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$

5



15 omitted

5 pts

Solve as TU game

+ 10

$$A-B = \begin{pmatrix} -4 & 0 & -3 \\ 2 & 1 & -1 \\ 5 & -1 & -3 \end{pmatrix}$$

$$\to \begin{pmatrix} -4 & -3 \\ 2 & -1 \\ 5 & -3 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}$$

$\delta = -1 \quad \sigma = 6$

$\varphi = \left(\frac{6-1}{2}, \frac{6+1}{2}\right) = \left(\frac{5}{2}, \frac{7}{2}\right)$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 1 & \boxed{1} \\ 5 & 2 & 0 \end{pmatrix} \qquad -B = \begin{pmatrix} -5 & -2 & -3 \\ -2 & 0 & \boxed{-2} \\ 0 & -3 & -3 \end{pmatrix}$$

Solve using Nash bargaining sol.

~~+15~~ OK

$\lambda^* =$ slope from $(1,2)$ to $\left(\frac{5}{2}, \frac{7}{2}\right)$

$= 1$

$= -$slope from $(1,5)$ to $(4,2)$

~~5 pts~~

Fixed threat point at $\langle 2, 3 \rangle$

Hence $\delta(\lambda) = \lambda - 2$.

5 pts

$$\sigma(\lambda) = \max(\lambda a_{ij} + b_{ij})$$

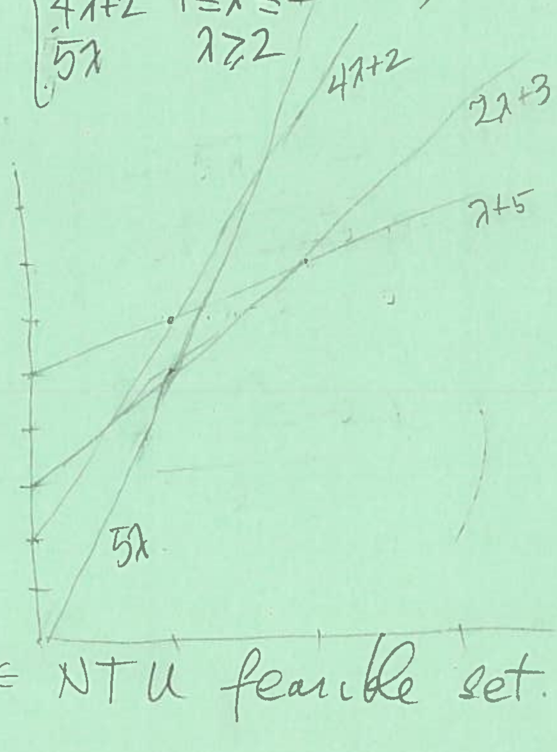
$$= \max(\lambda+5, \cancel{2\lambda+2}, \cancel{3}, 4\lambda+2, \cancel{\lambda}, \cancel{\lambda+2}, 5\lambda, 2\lambda+3, 3)$$

3 pts

$$= \max(\lambda+5, 2\lambda+3, 4\lambda+2, 5\lambda) = \begin{cases} \lambda+5 & \lambda \le 1 \\ 4\lambda+2 & 1 \le \lambda \le 2 \\ 5\lambda & \lambda \ge 2 \end{cases}$$

$\lambda+5 = 4\lambda+2 \Rightarrow \lambda = 1$

$4\lambda+5 = 5\lambda \Rightarrow \lambda = 2$



4 pts

$$\phi(\lambda) = \left(\frac{\sigma(\lambda) + \lambda - 2}{2\lambda}, \frac{\sigma(\lambda) - (\lambda - 2)}{2}\right)$$

$$\frac{\lambda+5+\lambda-2}{2\lambda} = \frac{2\lambda+3}{2\lambda} = 1 + \frac{3}{2\lambda}$$

3 pts.

With $\boxed{\lambda = 1}$ we get

$$\phi(1) = \left(\frac{6-1}{2}, \frac{6+1}{2}\right) = \left(\frac{5}{2}, \frac{7}{2}\right) \in \text{NTU feasible set.}$$

(6) players 1, 2, 3, 4, 5 10 omitted

$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$

$v(S) = 1$ if S contains 1 & 2

or 1 and not 2 and at least two of 3–5

or 2 and not 1 " " " " " "

$v(S) = 0$ otherwise. 8 pts.

Can omit
Then 10 pts each
for ϕ_1, ϕ_3

ϕ_3 has terms $\{1,3,4\}, \{1,3,5\}$
$\{2,3,4\}, \{2,3,5\}$

$$\phi_3(v) = \frac{2!\,2!}{5!} \cdot 4 = \boxed{\frac{2}{15}}$$ 6 pts

ϕ_1 has terms $\{1,2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}$
$\{1,3,4\}, \{1,3,5\}, \{1,4,5\}$
$\{1,3,4,5\}$

$$\phi_1(v) = \frac{1!\,3!}{5!} + \frac{2!\,2!}{5!} \cdot 3 + \frac{2!\,2!}{5!} \cdot 3 + \frac{3!\,1!}{5!}$$

$$= \frac{6+12+12+6}{120} = \frac{36}{120} = \boxed{\frac{3}{10}}$$ 6 pts

Check: $\frac{3}{10} \cdot 2 + \frac{2}{15} \cdot 3 = \frac{3}{5} + \frac{2}{5} = 1$.