Math 5040/6810
Midterm Exam—with solutions
October 23, 2009

Choose any 3 of the 4 problems, and turn in solutions for those problems only. Do not attempt all 4 problems, even if you have time.

1. One version of the Weibull distribution has *distribution function*

$$F(x) = 1 - e^{-x^2/2}, \qquad x > 0.$$

Use the inverse transformation method to create an algorithm for simulating a random variable X with distribution function F. You may call a random number generator, which will produce a uniform $(0, 1)$ random variable.

Solution: Find $F^{-1}(u)$. $F(x) = u$ implies $1 - e^{-x^2/2} = u$ or $x^2 = -2\ln(1-u)$ or $F^{-1}(u) = x = \sqrt{-2\ln(1-u)}$. The answer is $X = F^{-1}(U) = \sqrt{-2\ln(1-U)}$, where U is uniform on $(0, 1)$. We can simplify the answer by replacing $1 - U$ by U, which does not affect the distribution.

2. (a) Use the rejection method (with $g(x) = 1$) to create an algorithm for simulating a random variable X with the beta density $f(x) = 42x^5(1-x)$ $(0 < x < 1)$. You may call a random number generator as many times as needed, with each call producing an independent uniform $(0, 1)$ random variable. Your algorithm should be explicit enough that a programmer could create usable code from it.

(b) What is the expected number of calls to the random number generator per iteration?

Solution: (a) Let $h(x) = f(x)/g(x) = 42x^5(1-x)$. Then $h'(x) = 42(5x^4 - 6x^5) = 42x^4(5 - 6x) = 0$ implies $x = 5/6$. So $c = h(5/6) = 42(5/6)^5(1/6)$ and $h(x)/c = (6^6/5^5)x^5(1-x)$.

Algorithm. 1. Generate random numbers U_1 and U_2.

2. If $U_1 \le (6^6/5^5)U_2^5(1-U_2)$, then set $X = U_2$, else go to step 1.

3. X has the required distribution.

(b) It takes c steps to get an acceptance, on average, hence $2c$ random numbers are required.

3. Seven boys are playing with a ball.

The first boy always throws it to the second boy.

The second boy is equally likely to throw it to the third or seventh boy.

The third boy keeps the ball if he gets it.

The fourth boy always throws it to the sixth boy.

The fifth boy is equally likely to throw it to the fourth, sixth, or seventh boy.

The sixth boy always throws it to the fourth boy.

The seventh boy is equally likely to throw it to the first or fourth boy.

(a) Give the transition matrix $\boldsymbol{P}$ for the resulting Markov chain in $S = \{1, 2, 3, 4, 5, 6, 7\}$. Make sure rows and columns are aligned.

(b) Identify the communication classes, and indicate which are recurrent and which are transient.

(c) If boy 7 has the ball initially, what is the expected number of times that boy 1 gets the ball? (*) (see below)

(d) If boy 7 has the ball initially, what is the probability that boy 3 eventually gets it? (*)

Solution: (a) $\boldsymbol{P}$ has a 1 in the $(1, 2)$, $(3, 3)$, $(4, 6)$, and $(6, 4)$ positions, $1/2$ in the $(2, 3)$ and $(2, 7)$ positions as well as the $(7, 1)$ and $(7, 4)$ positions, and $1/3$ in the $(5, 4)$, $(5, 6)$, and $(5, 7)$ positions.

(b) The communicating classes are $\{3\}$ and $\{4, 6\}$, which are recurrent, and $\{1, 2, 7\}$ and $\{5\}$, which are transient.

(c) Let $\boldsymbol{Q}$ be the 4×4 submatrix corresponding to the transient states, that is,

$$\begin{array}{c} \\ \boldsymbol{Q} = \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 7 \end{array} \\ \begin{pmatrix} & 1 & & \\ & & & 1/2 \\ & & & 1/3 \\ 1/2 & & & \end{pmatrix} \end{array}.$$

With $\boldsymbol{M} = (\boldsymbol{I} - \boldsymbol{Q})^{-1}$, the answer is $(\boldsymbol{M})_{7,1}$.

(d) Let

$$\begin{array}{c} \\ \boldsymbol{S} = \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 7 \end{array} \begin{array}{c} \begin{array}{cc} 3 & \text{4-6} \end{array} \\ \begin{pmatrix} & \\ 1/2 & \\ & 2/3 \\ & 1/2 \end{pmatrix} \end{array}.$$

Then the answer is $(\boldsymbol{MS})_{7,3}$.

4. The Ehrenfest Markov chain is a simple model of the exchange of heat or of gas molecules between two isolated bodies. Suppose we have two urns

labeled 1 and 2, and d balls labeled $1, 2, \ldots, d$. Each of the balls is either in urn 1 or in urn 2. A transition occurs as follows. An integer is chosen at random from $1, 2, \ldots, d$, and the ball labeled by that integer is removed from its urn and placed in the other urn. This procedure is repeated indefinitely with the selections being independent from trial to trial. Let X_n be the number of balls in urn 1 after the nth trial. Then $\{X_n\}$ is a Markov chain in $S = \{0, 1, \ldots, d\}$ with a transition matrix $\boldsymbol{P}$.

For simplicity, assume $d = 4$.

(a) Determine $\boldsymbol{P}$.

(b) Find a stationary distribution $\boldsymbol{\pi}$. Here (*) *does not apply.* (Hint: It will simplify the algebra to notice that the stationary distribution must be symmetric about 2, i.e., $\pi_0 = \pi_4$ and $\pi_1 = \pi_3$.)

(c) If urn 1 is initially empty, what is the expected number of trials before it is full for the first time? (*)

Solution: (a)

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} & 1 & & & \\ 1/4 & & 3/4 & & \\ & 1/2 & & 1/2 & \\ & & 3/4 & & 1/4 \\ & & & 1 & \end{pmatrix} \end{array}.$$

(b) Solve $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$ and $\boldsymbol{\pi}\mathbf{1} = 1$. With $x = \pi_0 = \pi_4$, $y = \pi_1 = \pi_3$, and $z = \pi_2$, we have $x = y/4$, $y = x + z/2$, $z = 3y/2$, and $2x + 2y + z = 1$. Thus, $2(y/4) + 2y + 3y/2 = 1$, hence $y = 1/4$, implying $x = 1/16$ and $z = 3/8$. Our stationary distribution is $\boldsymbol{\pi} = (1/16)(1, 4, 6, 4, 1)$, which is the binomial distribution with $n = 4$ and $p = 1/2$.

(c) Make state 4 absorbing, so we have

$$\boldsymbol{Q} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} & 1 & & \\ 1/4 & & 3/4 & \\ & 1/2 & & 1/2 \\ & & 3/4 & \end{pmatrix} \end{array}.$$

The answer is then $[(\boldsymbol{I} - \boldsymbol{Q})^{-1}\mathbf{1}]_0$.

(*) Matrix operations need not be carried out. Instead, if your answer is the $(3, 2)$ entry of $\boldsymbol{A}^{-1}$, for example, just write $(\boldsymbol{A}^{-1})_{3,2}$. Of course you must specify $\boldsymbol{A}$ completely.