Math 5010-1, Exam 1, Oct. 9, 2013 Name<u> Solutions </u>

Instructions: *Do any 4 of the 5 problems.* Leave answers in combinatorial form. Electronic devices are not allowed.

1. (25 pts.) In poker dice 5 dice are rolled.

(a) Suppose all 5 dice are of different colors. How many distinguishable outcomes are possible?

$$6^5$$

(b) How many of the outcomes in (a) have the rank of full house (xxxyy)?

$$6 \cdot 5 \cdot \binom{5}{3}$$

(c) Now suppose all 5 dice are of the same color (indistinguishable). How many distinguishable outcomes are possible? (These outcomes may not be equally likely, but we are not concerned with probabilities.)

$$\binom{5+6-1}{6-1}$$

(d) How many of the outcomes in (c) have the rank of full house (xxxyy)?

$$6 \cdot 5$$

2. (25 pts.) Two decks of cards, one with blue backs and one with green backs, are mixed together, for a total of 104 cards. A card is drawn at random. Let A be the event that it is an ace. Let B be the event that it has a blue back. Let C be the event that it is a club.

(a) Note that $P(A) = 8/104 = 1/13$. Find $P(B)$ and $P(C)$.

$$1/2 \quad \text{and} \quad 1/4$$

(b) Note that $P(A \cap B) = 4/104 = 1/26$. Find $P(A \cap C)$, $P(B \cap C)$, and $P(A \cap B \cap C)$.

$$1/52, \quad 1/8, \quad 1/104$$

(c) Are A, B, C independent? Are they disjoint?

Yes and No

(d) Find the probability that exactly one of the events A, B, and C occurs. (Use a Venn diagram.)

$$P(A\cap B^c\cap C^c)+P(A^c\cap B\cap C^c)+P(A^c\cap B^c\cap C) = \frac{1}{13}\,\frac{1}{2}\,\frac{3}{4}+\frac{12}{13}\,\frac{1}{2}\,\frac{3}{4}+\frac{12}{13}\,\frac{1}{2}\,\frac{1}{4} = \frac{51}{104}$$

3. (25 pts.) Find the probability that a (13-card) bridge hand contains A,K,Q,J of *some* suit (i.e., A,K,Q,J of clubs or A,K,Q,J of diamonds or A,K,Q,J of hearts or A,K,Q,J of spades). Use inclusion-exclusion.

Let E_i be the event that the hand has A,K,Q,J of suit i. Then

$$P(E_1 \cup E_2 \cup E_3 \cup E_4) = \binom{4}{1}P(E_1) - \binom{4}{2}P(E_1E_2) + \binom{4}{3}P(E_1E_2E_3)$$

$$= \binom{4}{1}\frac{\binom{4}{4}\binom{48}{9}}{\binom{52}{13}} - \binom{4}{2}\frac{\binom{8}{8}\binom{44}{5}}{\binom{52}{13}} + \binom{4}{3}\frac{\binom{12}{12}\binom{40}{1}}{\binom{52}{13}}$$

4. (25 pts.) Urn I contains 3 red balls and 9 black balls. Urn II contains 8 red balls and 4 black balls. Urn III contains 10 red balls and 2 black balls. A card is drawn from a standard deck of 52. If a face card (J, Q, K) is drawn, a ball is selected from Urn I. If an ace is drawn, a ball is selected from Urn II. If any other card is drawn, a ball is selected from Urn III.

(a) Find the probability that a red ball is selected.

$$\frac{12}{52}\cdot\frac{3}{12}+\frac{4}{52}\cdot\frac{8}{12}+\frac{36}{52}\cdot\frac{10}{12}=\frac{107}{156}$$

(b) Find the conditional probability that Urn I was the one from which a ball was selected, given that the ball selected was red.

$$\frac{12}{52}\cdot\frac{3}{12}\Big/\frac{107}{156}=\frac{9}{107}$$

5. (25 pts.) Players A, B, and C take turns tossing a fair coin, first A, then B, then C, then A, then B, then C, then A, and so on. The first to toss heads wins. Find $P(\text{A wins})$, $P(\text{B wins})$, and $P(\text{C wins})$. (They should add up to 1.)

A wins with probability

$$1/2 + (1/2)^4 + (1/2)^7 + \cdots = (1/2)[1 + (1/2)^3 + (1/2)^6 + \cdots]$$

B wins with probability

$$(1/2)^2 + (1/2)^5 + (1/2)^8 + \cdots = (1/2)^2[1 + (1/2)^3 + (1/2)^6 + \cdots]$$

C wins with probability

$$(1/2)^3 + (1/2)^6 + (1/2)^9 + \cdots = (1/2)^3[1 + (1/2)^3 + (1/2)^6 + \cdots]$$

The three probabilities are in proportions $4:2:1$, so they must be 4/7, 2/7, and 1/7.