§ 5.4: Uniform Continuity

Def: Let $f: D \to \mathbb{R}$. We say that f is uniformly continuous on D if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta$ and $x, y \in D$.

i.e. δ works for all x.

Note: If a function is uniformly continuous, then it is continuous. Also we can talk of a function which is continuous at a point, but uniform continuity is only for a function on a set. Functions are not uniformly continuous at a point.

Ex: $f(x) = 2x$ $\forall x \in \mathbb{R}$. Then given $\varepsilon > 0$ we want $|f(x) - f(y)| < \varepsilon$ whenever x is "close enough" to y.

$$|f(x) - f(y)| = |2x - 2y| = 2|x-y|$$

So let $\delta = \frac{\varepsilon}{2}$.

Then if $|x-y| < \delta$ we have

$$|f(x) - f(y)| = 2|x-y| < 2\delta = \varepsilon.$$

$\therefore f$ is uniformly continuous on $\mathbb{R}$.

Ex: $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

$|f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y|$, but as x and y get bigger x^2 and y^2 get even larger. We will prove f is not uniformly continuous on $\mathbb{R}$.

Let $\varepsilon = 1$. We must show that given any $\delta > 0$, $\exists\, x, y \in \mathbb{R}$ s.t. $|x-y| < \delta$ but $|f(x) - f(y)| \geq 1$. Let $y = x + \frac{\delta}{2}$. Then $|x - y| = |x - (x + \frac{\delta}{2})| = \frac{\delta}{2} < \delta$. So if we want

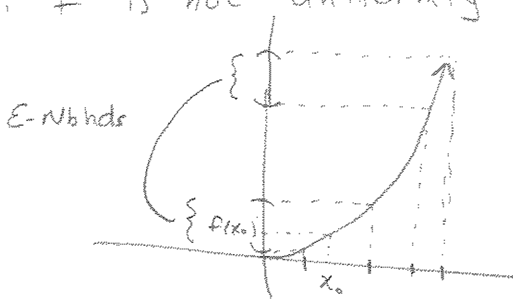
$$1 \leq |x+y||x-y| = |x+y| \cdot \frac{\delta}{2} \quad \text{we need} \quad |x+y| \geq \frac{2}{\delta}.$$

So let $x = \frac{1}{\delta}$.

Pf: Let $\varepsilon = 1$. Then given any $\delta > 0$, let $x = \frac{1}{\delta}$ and $y = \frac{1}{\delta} + \frac{\delta}{2}$. Then $|x-y| = |\frac{1}{\delta} - (\frac{1}{\delta} + \frac{\delta}{2})| = \frac{\delta}{2} < \delta$, but

$$|f(x) - f(y)| = |x+y||x-y| = \left|\frac{1}{\delta} + \frac{1}{\delta} + \frac{\delta}{2}\right| \cdot \frac{\delta}{2} > \frac{2}{\delta} \cdot \frac{\delta}{2} = 1.$$

$\therefore$ f is not uniformly continuous on $\mathbb{R}$. $\blacksquare$



Ex: The problem with the previous example is $|x+y|$ could be arbitrarily large. If we restrict our domain to a compact set, then this can't happen.

For example if $D = [-5, 5]$ $|x+y| \leq 10$. Thus given $\varepsilon > 0$, if $\delta = \frac{\varepsilon}{10}$ and $|x-y| < \delta$ we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y| < 10\delta = \varepsilon.$$

Thus f is uniformly continuous on D.

Theorem 76: Suppose that $f: D \to \mathbb{R}$ is continuous on a compact set D. Then f is uniformly continuous on D.

Pf: Let $\varepsilon > 0$ be given. Since f is continuous on D, f is continuous at every $x \in D$. Thus for each $x \in D$, $\exists\, \delta_x > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{2}$ whenever $|x-y| < \delta_x$, and $y \in D$.

Then the family of Nbhds $\mathcal{F} = \{N(x; \frac{\delta_x}{2}) : x \in D\}$ is an open cover of D. Since D is compact, $\mathcal{F}$ contains a finite subcover. That is, $\exists\, x_1, \ldots, x_n \in D$ s.t.

$$D \subseteq N(x_1; \tfrac{\delta_{x_1}}{2}) \cup \ldots \cup N(x_n; \tfrac{\delta_{x_n}}{2}).$$

Let $\delta = \min\{\frac{\delta_{x_1}}{2}, \ldots, \frac{\delta_{x_n}}{2}\}$. We claim this δ will work to show f is uniformly continuous.

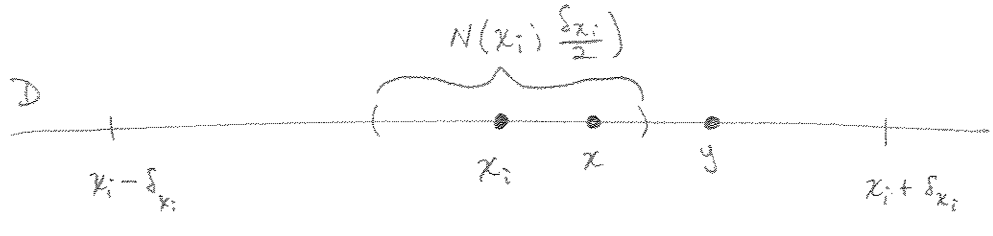
Suppose $x, y \in D$ and $|x-y| < \delta$, then $x \in D(x_i; \frac{\delta_{x_i}}{2})$ for some $i = 1, \ldots, n$. Since $|x-y| < \delta < \frac{\delta_{x_i}}{2}$, we have

$$|y - x_i| \le |y-x| + |x - x_i| < \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i}$$

Thus $|f(y) - f(x_i)| < \frac{\varepsilon}{2}$. But we also have $|x - x_i| < \frac{\delta_{x_i}}{2} < \delta_{x_i}$, So $|f(x_i) - f(x)| < \frac{\varepsilon}{2}$ $\therefore$

$$|f(x) - f(y)| \le |f(x_i) - f(x)| + |f(x_i) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

∎



Ex: Find a continuous function $f: D \to \mathbb{R}$ and a Cauchy sequence (x_n) in D s.t. $(f(x_n))$ diverges.

Let $f: (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. Then f is continuous. Let $x_n = \frac{1}{n}$. Then $f(x_n) = n$ which diverges.

Theorem 77: Let $f: D \to \mathbb{R}$ be uniformly continuous on D and suppose that (x_n) is a Cauchy sequence in D. Then $(f(x_n))$ is a Cauchy sequence.

Pf: Let $\varepsilon > 0$. Since f is uniformly continuous on D, $\exists\, \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$, $x, y \in D$. Since (x_n) is Cauchy, $\exists\, N$ s.t. $|x_n - x_m| < \delta$ whenever $n, m > N$. Thus for $n, m > N$ we have $|f(x_n) - f(x_m)| < \varepsilon$, so $(f(x_n))$ is a Cauchy sequence.

Def: We say a function $\tilde{f}: E \to \mathbb{R}$ is an <u>extension of a function</u> $f: D \to \mathbb{R}$ if $D \subseteq E$ and $f(x) = \tilde{f}(x)$ $\forall x \in D$.

Theorem 78: A function $f: (a,b) \to \mathbb{R}$ is uniformly continuous on (a,b) iff it can be extended to a function $\tilde{f}$ that is continuous on $[a,b]$.

Pf: If f can be extended to a function $\tilde{f}$ that is continuous on the compact set $[a,b]$, then $\tilde{f}$ is uniformly continuous on $[a,b]$ by Theorem <u>76</u>. It follows that $\tilde{f}$ (hence f) is also uniformly continuous on the subset (a,b).

Conversely, suppose f is uniformly continuous on (a,b). We claim $\lim_{x \to a} f(x)$ and $\lim_{x \to b} f(x)$ both exist as real numbers. To see this, let (s_n) be a sequence in (a,b) that converges to a. Then (s_n) is a Cauchy sequence, so Theorem <u>77</u> $\Rightarrow$ $(f(s_n))$ is also Cauchy. Theorem <u>60</u> $\Rightarrow (f(s_n)) \to p \in \mathbb{R}$. It follows from the contrapositive of Theorem 67 that $\lim_{x \to a} f(x) = p$. Similarly we can conclude $\lim_{x \to b} f(x) = q \in \mathbb{R}$.

Define $\tilde{f}:[a,b] \longrightarrow \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } a < x < b \\ p & \text{if } x = a \\ q & \text{if } x = b. \end{cases}$$

Then $\tilde{f}$ is an extension of f, and $\tilde{f}$ is continuous on $[a,b]$. ∎

Ex: Use the Theorem to determine if $\sin\left(\frac{1}{x}\right)$ is uniformly continuous on $\left(0, \frac{1}{\pi}\right)$.

Let $f(x) = \sin\left(\frac{1}{x}\right)$. f is continuous on $\left(0, \frac{1}{\pi}\right)$, but $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist, so f can not be extended to a continuous function on $\left[0, \frac{1}{\pi}\right] \Rightarrow f$ is not uniformly continuous on $\left(0, \frac{1}{\pi}\right)$.

Note: $g(x) = x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $g(0) = 0$ is u.c. on $\mathbb{R}$

$h(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$; $h(0) = 0$ is not u.c. on $\mathbb{R}$.

Theorem: Let f be a continuous function on an interval I, and let I° be the interval obtained by removing from I any endpoints of I. If f is diff'ble on I° and if f' is bounded on I°, then f is unif. cont. on I.

of later