§ 3.4: Topology of the Reals

Much of analysis deals with the idea of two points being "close" to each other. The distance between two points $x, y \in \mathbb{R}$ is $|x-y|$. Thus if we have some positive measure of closeness, say ε, we may be interested in all the points $y \in \mathbb{R}$ that are less then ε away from x, i.e. $\{y : |x-y| < \varepsilon\}$

Def: Let $x \in \mathbb{R}$ and $\varepsilon > 0$. A <u>Nbhd</u> of x (or an ε-neighborhood ~~of~~ x) is a set of the form

$$N(x;\varepsilon) = \{y \in \mathbb{R} \mid |x-y| < \varepsilon\}.$$

ε is the <u>radius</u> of $N(x;\varepsilon)$.

Basically $N(x;\varepsilon) = (x-\varepsilon, x+\varepsilon)$. We used "neighorhood" instead of interval because it is more general. With the idea of nbhd's, we can define concepts of open & closed ~~sets~~.
The study of open and closed sets is called <u>point set topology</u>

In some situations, we want to consider points close to x, but different from x. So we consider $|x-y| > 0$.

Def: Let $x \in \mathbb{R}$ and $\varepsilon > 0$. A <u>deleted Nbhd</u> of x is a set of the form

$$N^*(x;\varepsilon) = \{y \in \mathbb{R} \mid 0 < |x-y| < \varepsilon\}.$$

If $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, either x is "inside S", x is on the "edge" of S, or x is "outside" S.
Using Nbhds we can make these notions precise.
Def: Let $S \subseteq \mathbb{R}$. $x \in \mathbb{R}$ is an <u>interior</u> point of S if $\exists$ a nbhd N of x s.t. $N \subseteq S$. If every nbhd N of x satisfies $N \cap S \neq \emptyset$ and $N \cap (\mathbb{R} \setminus S) \neq \emptyset$, then x is a <u>boundary</u> point of S. The set of all interior points of S is denoted $\operatorname{int} S$, boundary points $\operatorname{bd} S$.
Note: x "outside" $S \Rightarrow x$ is an interior point of $\mathbb{R} \setminus S$.

$\varepsilon \geq 5-x$
$\varepsilon \geq x \Rightarrow -x \leq -\varepsilon$

Ex: a) Let $S=(0,5) \subseteq \mathbb{R}$, And let $x \in S$. If $\varepsilon = \min\{x, 5-x\}$ then we claim $N(x;\varepsilon) \subseteq S$. Indeed $\forall y \in N(x;\varepsilon)$ we have $|y-x|<\varepsilon$ so $-x \leq -\varepsilon < y-x < \varepsilon \leq 5-x$.

Thus $0<y<5$ and $y \in S$. It follows that $\forall x \in S$, $x \in \operatorname{int} S$.

Since $\operatorname{int} S \subseteq S$ always holds, we have $S = \operatorname{int} S$.

$0 \notin S$, but every nbhd of 0 will contain elements in S, So $0 \in \operatorname{bd} S$. Similarly $5 \in \operatorname{bd} S$. In fact $\operatorname{bd} S = \{0,5\}$.

b) Let $R=[0,5]$.

$0 \in \operatorname{bd} R$ since every nbhd of 0 contains elements not in R

So $\operatorname{int} R = (0,5)$, $\operatorname{bd} R = \{0,5\}$, and $\operatorname{bd} R \subseteq R$.

c) $T = [0,5)$ $\quad \operatorname{int} T = (0,5)$ $\quad \operatorname{bd} T = \{0,5\}$

but $\operatorname{bd} T \nsubseteq T$.

Def: Let $S \subseteq \mathbb{R}$. If $\operatorname{bd} S \subseteq S$, then S is closed. If $\operatorname{bd} S \subseteq \mathbb{R} \setminus S$ then S is open.

ies.

Theorem 41: a) A set S is open iff $S = \operatorname{int} S$. Equivalently, S is open iff every point in S is an interior point of S.

b) A set S is closed iff its complement is open.

Ex: $(0,5)$ is open $\quad [0,5]$ is closed, so notion already works.

$[0,5)$ is neither open nor closed.

$[2,\infty)$ is closed.

Q: Is $\mathbb{R}$ open or closed? $\operatorname{int} \mathbb{R} = \mathbb{R}$, so $\mathbb{R}$ is open.

$\operatorname{bd} \mathbb{R} = \emptyset$ and $\emptyset \subseteq \mathbb{R}$, so $\mathbb{R}$ is closed.

Q: Is $\emptyset$ open or closed?

Theorem 42:

a) The union of any collection of open sets is open.

b) The intersection of any finite collection of open sets is an open set.

Pf: a) Let $\mathcal{O}$ be an arbitrary collection of open sets, and let $S = \bigcup \mathcal{O}$. If $x \in S$, then $x \in A$ for some $A \in \mathcal{O}$. Since A is open $\exists N$ of x s.t. $N \subseteq A$, $A \subseteq S \Rightarrow N \subseteq S \Rightarrow x \in \text{int } S$.

$\therefore S$ is open. ☒

b) Let $A_1, \dots, A_n$ be a finite collection of open sets. Let $T = \bigcap_{i=1}^{n} A_i$. If $T = \emptyset$ we are done since $\emptyset$ is open. If $T \neq \emptyset$, let $x \in T$. Then $x \in A_i$ $\forall i = 1, \dots, n$. Since each A_i is open $\exists N_i(x; \varepsilon_i)$ neighborhoods of x s.t. $N_i \subseteq A_i$. Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$. Then $N(x;\varepsilon) \subseteq A_i$ for all $i = 1, \dots, n \Rightarrow N(x;\varepsilon) \subseteq T \Rightarrow x \in \text{int } T$. So T is open. ☒

Corollary 11: a) The intersection of any collection of closed sets is closed.

b) The finite union of any collection of closed sets is closed.

Pf: Both parts follow from Thrm 42 combined with Thrm 41. Recall from HW 4 problem 1, that $\mathbb{R} \setminus \left(\bigcup_{j \in J} A_j\right) = \bigcap_{j \in J} (\mathbb{R} \setminus A_j)$ and $\mathbb{R} \setminus \left(\bigcap_{j \in J} A_j\right) = \bigcup_{j \in J} (\mathbb{R} \setminus A_j)$. ☒

Why is there the restriction to finitely many sets?

$A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$ = closed set.

Accumulation Points

Def: Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is an accumulation point of S if every deleted Nbhd of x contains a point of S. i.e. $\forall \varepsilon > 0$ $N^*(x;\varepsilon) \cap S \neq \emptyset$.

We denote the set of all accumulation points of S by S'.

If $x \in S$ and $x \notin S'$, x is an isolated point of S.

Ex: a) $S = (0,1]$. $S' = [0,1]$.

b) $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, $S' = \{0\}$.

c) $S = \mathbb{N}$, $S' = \emptyset$ thus $\mathbb{N}$ consists entirely of isolated points

d) If S is a finite set, then $S' = \emptyset$. Indeed, if $S = \{x_1, \dots, x_n\}$ and $y \in \mathbb{R}$, let $\varepsilon = \min\{|x_i - y| : x_i \neq y\}$. Then $\varepsilon > 0$ and $N^*(y;\varepsilon) \cap S = \emptyset$. So y is not an accumulation pt. of S.

Def: Let $S \subseteq \mathbb{R}$. Then the closure of S, denoted $\text{cl}\, S$, is

$$\text{cl}\, S = S \cup S'$$

i.e. $x \in \text{cl}\, S \iff$ every Nbhd of x intersects S.

Theorem 43: Let $S \subseteq \mathbb{R}$. Then

a) S is closed $\iff$ S contains all of its accumulation points

b) $\text{cl}\, S$ is a closed set.

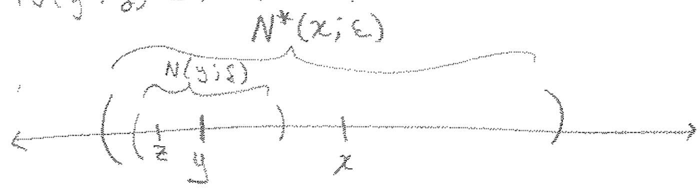
c) S is closed $\iff$ $S = \text{cl}\, S$.

Pf: a) Suppose S is closed and let $x \in S'$. We must show $x \in S$. If $x \notin S$ then $x \in \mathbb{R} \setminus S$ which is open. Thus $\exists$ Nbhd N of x s.t. $N \subseteq \mathbb{R} \setminus S$ $\Rightarrow N \cap S = \emptyset$ which contradicts $x \in S'$.

Conversely suppose $S' \subseteq S$. We will show $\mathbb{R} \setminus S$ is open. Let $x \in \mathbb{R} \setminus S$. Then $x \notin S'$ so $\exists$ $N^*(x;\varepsilon)$ s.t. $N^*(x;\varepsilon) \cap S = \emptyset$, Since $x \notin S$. So $N(x;\varepsilon) \subseteq \mathbb{R} \setminus S$, thus $\mathbb{R} \setminus S$ is open $\Rightarrow$ S is closed. $\boxtimes$

b) By part (a) we must show if $x \in (\text{cl } S)'$, then $x \in \text{cl } S$. So suppose x is an accumulation pt of $\text{cl } S$. Then every deleted Nbhd $N^*(x;\varepsilon)$ intersects $\text{cl } S$. we must show $N^*(x;\varepsilon) \cap S \neq \emptyset$. Let $y \in N^*(x;\varepsilon) \cap \text{cl } S$. Since $N^*(x;\varepsilon)$ is open, $\exists$ a nbhd $N(y;\delta)$ contained in $N^*(x,\varepsilon)$. But $y \in \text{cl } S$ so $N(y;\delta) \cap S \neq \emptyset$. So $\exists\, z \in N(y;\delta) \cap S$. But

$z \in N(y;\delta) \subseteq N^*(x,\varepsilon)$, so $x \in S'$ and $x \in \text{cl } S$. ∎



(c) Exercise.