Chapter 1

Logical Connectives and Quantifiers

1.1 Logical Connectives

1.2 Quantifiers

1.3 Techniques of Proof: I

1.4 Techniques of Proof: II

Theorem 1. *Let* f *be a continuous function. If* $\int_0^1 f(x)\,dx \neq 0$*, then there exists a point* x *in the interval* $[0,1]$ *such that* $f(x) \neq 0$*.*

Theorem 2. *Let* x *be a real number. If* $x > 0$*, then* $\frac{1}{x} > 0$*.*

Theorem 3. *If the sum of a real number with itself is equal to its square, then the number is 0 or 2.*

Chapter 2

Sets and Functions

2.1 Basic Set Operations

Theorem 4. *Let* A *be a set. Then* $\emptyset \subseteq A$.

Theorem 5. *Let* A *and* B *be subsets of a universal set* U. *Then* $A \cap (U \setminus B) = A \setminus B$.

Theorem 6. *Let* A, B, *and* C *be subsets of a universal set* U. *Then the following statements are true.*

1. $A \cup (U \setminus A) = U$.
2. $A \cap (U \setminus A) = \emptyset$.
3. $U \setminus (U \setminus A) = A$.
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
6. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.
7. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Theorem 7. *If* A *and* B *are subsets of a set* U *and* A^c *and* B^c *are their complements in* U, *then*

1. $(A \cup B)^c = A^c \cap B^c$.
2. $(A \cap B)^c = A^c \cup B^c$.

2.2 Relations

Theorem 8. *$(a,b)=(c,d)$ iff $a=c$ and $b=d$.*

Theorem 9. *Let R be an equivalence relation on a set S. Then $\{E_x : x \in S\}$ is a partition of S. The relation "belongs to the same piece as" is the same as R. Conversely, if $\mathcal{T}$ is a partition of S, let R be defined by xRy iff x and y are in the same piece of the partition. Then R is an equivalence relation and the corresponding partition into equivalence classes is the same as $\mathcal{T}$.*

2.3 Functions

Theorem 10. *Suppose that $f : A \to B$. Let C, C_1 and C_2 be subsets of A and let D, D_1 and D_2 be subsets of B. Then the following hold:*

1. $C \subseteq f^{-1}[f(C)]$.
2. $f[f^{-1}(D)] \subseteq D$.
3. $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$.
4. $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.
5. $f(C_1) \setminus f(C_2) \subseteq f(C_1 \setminus C_2)$ *if* $C_2 \subseteq C_1$.
6. $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$.
7. $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$.
8. $f^{-1}(B \setminus D) = A \setminus f^{-1}(D)$.
9. $f^{-1}(D_1 \setminus D_2) = f^{-1}(D_1) \setminus f^{-1}(D_2)$ *if* $D_2 \subseteq D_1$.

Theorem 11. *Suppose that $f : A \to B$. Let C, C_1 and C_2 be subsets of A and let D be a subset of B. Then the following hold:*

1. *If f is injective, then $f{-}1[f(C)] = C$.*
2. *If f is surjective, then $f[f^{-1}(D)] = D$.*
3. *If f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.*

Theorem 12. *Let $f : A \to B$ and $g : B \to C$. Then*

1. *If f and g are surjective, then $g \circ f$ is surjective.*
2. *If f and g are injective, then $g \circ f$ is injective.*
3. *If f and g are bijective, then $g \circ f$ is bijective.*

Theorem 13. *Let $f : A \to B$ be bijective. Then*

1. *$f^{-1} : B \to A$ is bijective.*

2. $f^{-1} \circ f = i_A$ *and* $f \circ f^{-1} = i_B$.

Theorem 14. *Let* $f : A \to B$ *and* $g : B \to C$ *be bijective. The the composition* $g \circ f : A \to C$ *is bijective and* $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

2.4 Cardinality

Theorem 15. *Let* S *be a countable set and let* $T \subseteq S$. *Then* T *is countable.*

Theorem 16. *Let* S *be a nonempty set. The following three conditions are equivalent:*

1. S *is countable.*
2. *There exists an injection* $f : S \to \mathbb{N}$.
3. *There exists a surjection* $f : \mathbb{N} \to S$.

Theorem 17. *The set* $\mathbb{R}$ *of real numbers is uncountable.*

Theorem 18. *Let* S, T *and* U *be sets.*

1. *If* $S \subseteq T$, *then* $|S| \leq |T|$.
2. $|S| \leq |S|$.
3. *If* $|S| \leq |T|$ *and* $|T| \leq |U|$, *then* $|S| \leq |U|$.
4. *If* $m, n \in \mathbb{N}$ *and* $m \leq n$, *then* $|\{1, 2, \ldots, m\}| \leq |\{1, 2, \ldots, n\}|$.
5. *If* S *is finite, then* $S < \aleph_0$.

Theorem 19. *For any set* S, *we have* $|S| < |\mathcal{P}(\mathcal{S})|$.

Chapter 3

The Real Numbers

3.1 Natural Numbers and Induction

Axiom 1. *(Well-Ordering Property of $\mathbb{N}$) If S is a nonempty subset of $\mathbb{N}$, then there exists an element $m \in S$ such that $m \leq k$ for all $k \in S$.*

Theorem 20. *(Principle of Mathematical Induction) Let $P(n)$ be a statement that is either true or false for each $n \in \mathbb{N}$. Then $P(n)$ is true for all $n \in \mathbb{N}$ provided that*

1. *$P(1)$ is true, and*
2. *for each $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.*

Theorem 21. $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$ *for every natural number n.*

Theorem 22. $7^n - 4^n$ *is a multiple of 3 for all $n \in \mathbb{N}$.*

Theorem 23. *(The binomial formula) If x and y are real numbers and $n \in \mathbb{N}$, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{x}{y} x^k y^{n-k}.$$

Theorem 24. *Let $m \in \mathbb{N}$ and let $P(n)$ be a statement that is either true or false for each $n \geq m$. Then $P(n)$ is true for all $n \geq m$ provided that*

1. *$P(m)$ is true, and*
2. *for each $k \geq m$, if $P(k)$ is true, then $P(k+1)$ is true.*

3.2 Ordered Fields

Theorem 25. *Let x, y, and z be real numbers.*

1. *If $x + z = y + z$, then $x = y$.*

2. *$x \cdot 0 = 0$.*
3. *$(-1) \cdot x = -x$.*
4. *$xy = 0$ iff $x = 0$ or $y = 0$.*
5. *$x < y$ iff $-y < -x$.*
6. *If $x < y$ and $z < 0$, then $xz > yz$.*

Theorem 26. *Let $x, y \in \mathbb{R}$ such that $x \leq y + \epsilon$ for every $\epsilon > 0$. Then $x \leq y$.*

Theorem 27. *Let $x, y \in \mathbb{R}$ and let $a \geq 0$. Then*

1. *$|x| \geq 0$.*
2. *$|x| \leq a$ iff $-a \leq x \leq a$.*
3. *$|xy| = |x| \cdot |y|$.*
4. *$|x + y| \leq |x| + |y|$. (The triangle inequality)*

3.3 The Completeness Axiom

Theorem 28. *Let $m, n, p \in \mathbb{Z}$. If p is a prime number and p divides the product mn, then p divides m or p divides n.*

Theorem 29. *Let p be a prime number. Then $\sqrt{p}$ is not a rational number.*

Axiom 2. *(The completeness axiom) Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists and is a real number.*

Theorem 30. *Every non-empty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.*

Theorem 31. *Let A be a non-empty subset of $\mathbb{R}$ and x an element of $\mathbb{R}$. Then*

1. *$\sup A \leq x$ iff $a \leq x$ for every $a \in A$.*
2. *$x < \sup A$ iff $x < a$ for some $a \in A$.*

Theorem 32. *Let A and B be non-empty subsets of $\mathbb{R}$. Then*

1. *$\inf A \leq \sup A$.*
2. *$\sup(-A) = -\inf A$ and $\inf(-A) = -\sup A$.*
3. *$\sup(A + B) = \sup(A) + \sup(B)$ and $\inf(A + B) = \inf(A) + \inf(B)$.*
4. *$\sup(A - B) = \sup(A) - \inf(B)$.*
5. *If $A \subseteq B$, then $\sup A \leq \sup B$ and $\inf B \leq \inf A$.*

Theorem 33. *Suppose that D is a nonempty set and that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. If for every $x, y \in D$, $f(x) \leq g(y)$, then $f(D)$ is bounded above and $g(D)$ is bounded below. Furthermore,* $\sup f(D) \leq \inf g(D)$.

Theorem 34. *Let f and g be functions defined on a set containing A as a subset, and let $c \in \mathbb{R}$ be a positive constant. Then*

1. $\sup_A cf = c \sup_A f$ *and* $\inf_A cf = c \inf_A f$.
2. $\sup_A(-f) = -\inf_A f$.
3. $\sup_A(f+g) \leq \sup_A f + \sup_A g$ *and* $\inf_A f + inf_A g \leq \inf_A(f+g)$.
4. $\sup\{f(x) - f(y) : x, y \in A\} \leq \sup_A f - \inf_A f$.

Theorem 35. *The real number system $\mathbb{R}$ is a complete ordered field.*

Theorem 36. *(Archimedean Property of $\mathbb{R}$) The set $\mathbb{N}$ of natural numbers is unbounded above in $\mathbb{R}$.*

Theorem 37. *Each of the following is equivalent to the Archimedean property.*

1. *For each $z \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > z$.*
2. *For each $x > 0$ and for each $y \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $nx > y$.*
3. *For each $x > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$.*

Theorem 38. *Let p be a prime number. Then there exists a positive real number x such that $x^2 = p$.*

Theorem 39. *(Density of $\mathbb{Q}$ in $\mathbb{R}$) If x and y are real numbers with $x < y$, then there exists a rational number r such that $x < r < y$.*

Theorem 40. *If x and y are real numbers with $x < y$, then there exists an irrational number w such that $x < w < y$.*

3.4 Topology of the Reals

Theorem 41.
1. *A set S is open iff $S = int\, S$. Equivalently, S is open iff every point in S is an interior point of S.*
2. *A set S is closed iff its complement $\mathbb{R} \setminus S$ is open.*

Theorem 42.
1. *The union of any collection of open sets is an open set.*
2. *The intersection of any finite collection of open sets is an open set.*

Corollary 1.
1. *The intersection of any collection of closed sets is closed.*
2. *The union of any finite collection of closed sets is closed.*

Theorem 43. *Let S be a subset of $\mathbb{R}$. Then*

1. *S is closed iff S contains all of its accumulation points.*
2. *cl S is a closed set.*
3. *S is closed iff $S =$ cl S.*

3.5 Compact Sets

Lemma 1. *If S is a nonempty closed bounded subset of $\mathbb{R}$, then S has a maximum and a minimum.*

Theorem 44. *(Heine-Borel) A subset S of $\mathbb{R}$ is compact iff S is closed and bounded.*

Theorem 45. *(Bolzano-Weierstrass) If a bounded subset S of $\mathbb{R}$ contains infinitely many points, then there exists at least one point in $\mathbb{R}$ that is an accumulation point of S.*

Theorem 46. *Let $\mathcal{F} = \{\mathcal{K}_\alpha : \alpha \in \mathcal{A}\}$ be a family of compact subsets of $\mathbb{R}$. Suppose that the intersection of any finite subfamily of $\mathcal{F}$ is nonempty. Then $\bigcap\{K_\alpha : \alpha \in \mathcal{A}\} \neq \emptyset$.*

Corollary 2. *(Nested Intervals Theorem) Let $\mathcal{F} = \{A_n : n \in \mathbb{N}\}$ be a family of closed bounded intervals in $\mathbb{R}$ such that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.*

Chapter 4

Sequences

4.1 Convergence

Theorem 47. *Let* (s_n) *and* (a_n) *be sequences of real numbers and let* $s \in \mathbb{R}$*. If for some* $k > 0$ *and some* $m \in \mathbb{N}$*, we have*

$$|s_n - s| \leq k|a_n|, \text{ for all } n > m,$$

and if $\lim a_n = 0$*, then it follows that* $\lim s_n = s$*.*

Theorem 48. *Every convergent sequence is bounded.*

Theorem 49. *If a sequence converges, its limit is unique.*

Theorem 50. *A sequence* (s_n) *converges to* s *iff for each* $\epsilon > 0$*, there are only finitely many* n *for which* $|s_n - s| \geq \epsilon$*.*

Theorem 51. *Let* (s_n) *be a sequence of real numbers such that* $\lim s_n = 0$*, and let* (t_n) *be a bounded sequence. Then* $\lim s_n t_n = 0$*.*

Theorem 52. *(The squeeze principle) If* (a_n)*,* (b_n)*, and* (c_n) *are sequences for which there is a number* K *such that*

$$b_n \leq a_n \leq c_n \text{ for all } n > K,$$

and if $b_n \to a$ *and* $c_n \to a$*, then* $a_n \to a$*.*

4.2 Limit Theorems

Theorem 53. *Suppose that* (s_n) *and* (t_n) *are convergent sequences with* $\lim s_n = s$ *and* $\lim t_n = t$*. Then*

1. $\lim(s_n + t_n) = s + t$.
2. $\lim(ks_n) = ks$ *and* $\lim(k + s_n) = k + s$ *for any* $k \in \mathbb{R}$.

3. *$\lim(s_n t_n) = st$.*

4. *$\lim\left(\frac{s_n}{t_n}\right) = \frac{s}{t}$, provided that $t_n \neq 0$ for all n and $t \neq 0$.*

Theorem 54. *Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$ and $\lim t_n = t$. If $s_n \leq t_n$ for all $n \in \mathbb{N}$, then $s \leq t$.*

Corollary 3. *If (t_n) converges to t and $t_n \geq 0$ for all $n \in \mathbb{N}$, then $t \geq 0$.*

Theorem 55. *(Ratio Test) Suppose that (s_n) is a sequence of positive terms and that the limit $L = \lim\left(\frac{s_{n+1}}{s_n}\right)$ exists. If $L < 1$, then $\lim s_n = 0$.*

Theorem 56. *Suppose that (s_n) and (t_n) are sequences such that $s_n \leq t_n$ for all $n \in \mathbb{N}$.*

1. *If $\lim s_n = +\infty$, then $\lim t_n = +\infty$.*

2. *If $\lim t_n = -\infty$, then $\lim s_n = -\infty$.*

Theorem 57. *Let (s_n) be a sequence of positive numbers. Then $\lim s_n = +\infty$ iff $\lim\left(\frac{1}{s_n}\right) = 0$.*

4.3 Monotone Sequences and Cauchy Sequences

Theorem 58. *(Monotone Convergence Theorem) A monotone sequence is convergent iff it is bounded.*

Theorem 59.

1. *If (s_n) is an unbounded increasing sequence, then $\lim s_n = +\infty$.*

2. *If (s_n) is an unbounded decreasing sequence, then $\lim s_n = -\infty$.*

Lemma 2. *Every convergent sequence is a Cauchy sequence.*

Lemma 3. *Every Cauchy sequence is bounded.*

Theorem 60. *(Cauchy Convergence Criterion) A sequence of real numbers is convergent iff it is a Cauchy sequence.*

4.4 Subsequences

Theorem 61. *If a sequence (s_n) converges to a real number s, then every subsequence of (s_n) also converges to s.*

Theorem 62. *(Bolzano-Weierstrass Theorem For Sequences) Every bounded sequence has a convergent subsequence.*

Theorem 63. *Every unbounded sequence contains a monotone subsequence that has either $+\infty$ or $-\infty$ as a limit.*

Theorem 64. *Let* (s_n) *be a sequence and suppose that* $m = \limsup s_n$ *is a real number. Then the following properties hold:*

1. *For every* $\epsilon > 0$ *there exists* N *such that* $n > N$ *implies that* $s_n < m + \epsilon$.
2. *For every* $\epsilon > 0$ *an for every* $i \in \mathbb{N}$, *there exists an integer* $k > i$ *such that* $s_k > m - \epsilon$.

Chapter 5

Limits and Continuity

5.1 Limits of Functions

Theorem 65. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x\to c} f(x) = L$ iff for each neighborhood V of L there exists a deleted neighborhood U of c such that $f(U \cap D) \subseteq V$.*

Theorem 66. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x\to c} f(x) = L$ iff for every sequence (s_n) in D that converges to c with $s_n \neq c$ for all n, the sequence $(f(s_n))$ converges to L.*

Corollary 4. *If $f : D \to \mathbb{R}$ and if c is an accumulation point of D, then f can have only one limit at c.*

Theorem 67. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then the following are equivalent:*

(a) f does not have a limit at c.

(b) There exists a sequence (s_n) in D with each $s_n \neq c$ such that (s_n) converges to c, but $(f(s_n))$ is not convergent in $\mathbb{R}$.

Theorem 68. *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$, and let c be an accumulation point of D. If $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$, and $k \in \mathbb{R}$, then $\lim_{x\to c}(f + g)(x) = L + M, \lim_{x\to c}(fg)(x) = LM$, and $\lim_{x\to c}(kf)(x) = kL$.*

5.2 Continuous Functions

Theorem 69. *Let $f : D \to \mathbb{R}$ and let $c \in D$. Then the following three conditions are equivalent:*

(a) f is continuous at c.

(b) If (x_n) is any sequence in D such that (x_n) converges to c, then $\lim f(x_n) = f(c)$.

(c) For every neighborhood V of $f(c)$ there exists a neighborhood U of c such that $f(U \cap D) \subseteq V$.

Furthermore, if c is an accumulation point of D, then the above are all equivalent to

(d) f has a limit at c and $\lim_{x\to c} f(x) = f(c)$.

Theorem 70. *Let $f : D \to \mathbb{R}$ and let $c \in D$. Then f is discontinuous at c iff there exists a sequence (x_n) in D such that (x_n) converges to c but the sequence $(f(x_n))$ does not converge to $f(c)$.*

Theorem 71. *Let f and g be functions from D to $\mathbb{R}$, and let $c \in D$. Suppose that f and g are continuous at c. Then*

(a) $f + g$ and fg are continuous at c,

(b) f/g is continuous at c if $g(c) \neq 0$.

Theorem 72. *Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$. If f is continuous at a point $c \in D$ and g is continuous at $f(c)$, then the composition $g \circ f : D \to \mathbb{R}$ is continuous at c.*

5.3 Properties of Continuous Functions

Theorem 73. *Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.*

Corollary 5. *Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then f assumes minimum and maximum values on D. That is, there exist points x_1 and x_2 in D such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in D$.*

Lemma 4. *Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a) < 0 < f(b)$. Then there exists a point c in (a, b) such that $f(x) = 0$.*

Theorem 74. *(Intermediate Value Theorem) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then f has the intermediate value property on $[a, b]$. That is, if k is any value between $f(a)$ and $f(b)$ [i.e. $f(x) < k < f(b)$ or $f(b) < k < f(a)$], then there exists $c \in [a, b]$ such that $f(c) = k$.*

Theorem 75. *Let I be a compact interval and suppose that $f : I \to \mathbb{R}$ is a continuous function. Then the set $f(I)$ is a compact interval.*

5.4 Uniform Continuity

Theorem 76. *Suppose that $f : D \to \mathbb{R}$ is continuous on a compact set D. Then f is uniformly continuous on D.*

Theorem 77. *Let* $f : D \to \mathbb{R}$ *be uniformly continuous on* D *and suppose that* (x_n) *is a Cauchy sequence in* D*. Then* $(f(x_n))$ *is a Cauchy sequence.*

Theorem 78. *A function* $f : (a, b) \to \mathbb{R}$ *is uniformly continuous on* (a, b) *iff it can be extended to a function* $\tilde{f}$ *that is continuous on* $[a, b]$.

Chapter 6

Differentiation

6.1 The Derivative

Theorem 79. *Let I be an interval containing the point c and suppose that $f : I \to \mathbb{R}$. Then f is differentiable at c iff, for every sequence (x_n) in $I \setminus \{c\}$ that converges to c, the sequence*

$$\left(\frac{f(x_n) - f(c)}{x_n - c}\right)$$

converges. Furthermore, if f is differentiable at c, then the sequence of quotients above will converge to $f'(c)$.

Theorem 80. *If $f : I \to \mathbb{R}$ is differentiable at a point $c \in I$, then f is continuous at c.*

Theorem 81. *Suppose that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$. Then*

- *(a) If $k \in \mathbb{R}$, then the function kf is differentiable at c and $(kf)'(c) = k \cdot f'(c)$.*
- *(b) The function $f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$.*
- *(c) (Product Rule) The function fg is differentiable at c and $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$.*
- *(d) (Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c and*

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}.$$

Theorem 82. *(Chain Rule) Let I and J be intervals in $\mathbb{R}$, let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, where $f(I) \subseteq J$, and let $c \in I$. If f i differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.*

6.2 The Mean Value Theorem

Theorem 83. *If f is differentiable on an open interval (a, b) and if f assumes its maximum or minimum at a point $c \in (a, b)$, then $f'(c) = 0$.*

Theorem 84. *(Rolle's Theorem) Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and such that $f(a) = f(b) = 0$. Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.*

Theorem 85. *(Mean Value Theorem) Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Theorem 86. *Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.*

Corollary 6. *Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there exists a constant C such that $f = g + C$ on $[a, b]$.*

Theorem 87. *Let f be differentiable on an interval I. Then*

(a) if $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on i, and

(b) if $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.

Theorem 88. *Let f be a continuous function on an interval I, and let I° be the interval obtained by removing from I any endpoints of I. If f is differentiable on I° and if f' is bounded on I°, then f is uniformly continuous on I.*

Theorem 89. *(Intermediate Value Theorem for Derivatives) Let f be differentiable on $[a, b]$ and suppose that k is a number between $f'(a)$ and $f'(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = k$.*

Theorem 90. *(Inverse Function Theorem) Suppose that f is differentiable on an interval I and $f'(x) \neq 0$ for all $x \in I$. Then f is injective, f^{-1} is differentiable on $f(I)$, and $(f^{-1})'(y) = \frac{1}{f'(x)}$, where $y = f(x)$.*

6.3 L'Hospital's Rule

Theorem 91. *(Cauchy Mean Value Theorem) Let f and g be functions that are continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one point $c \in (a, b)$ such that $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$.*

Theorem 92. *(L'Hospital's Rule) Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $c \in [a, b]$ an $f(c) = g(c) = 0$. Suppose also that $g'(x) \neq 0$ for $x \in U$, where U is the intersection of (a, b) and some deleted neighborhood of c. If $\displaystyle\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$, with $L \in \mathbb{R}$, then $\displaystyle\lim_{x\to c} \frac{f(x)}{g(x)} = L$.*

Theorem 93. *(L'Hospital's Rule) Let* f *and* g *be differentiable on* (b, ∞)*. Suppose that* $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$*, and that* $g'(x) \neq 0$ *for* $x \in (b, \infty)$*. If* $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$*, where* $L \in \mathbb{R}$*, then* $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$*.*

6.4 Taylor's Theorem

Theorem 94. *(Taylor's Theorem) Let* f *and its first* n *derivatives be continuous on* $[a, b]$ *and differentiable on* (a, b)*, and let* $x_0 \in [a, b]$*. Then for each* $x \in [a, b]$ *with* $x \neq x_0$ *there exists a point* c *between* x *and* x_0 *such that*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

$$+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Chapter 7

Integration

7.1 The Riemann Integral

Theorem 95. *Let* f *be a bounded function on* $[a, b]$. *If* P *and* Q *are partitions of* $[a, b]$ *and* Q *is a refinement of* P, *then* $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$.

Theorem 96. *Let* f *be a bounded function on* $[a, b]$. *Then* $L(f) \leq U(f)$.

Theorem 97. *Let* f *be a bounded function on* $[a, b]$. *Then* f *is integrable iff for each* $\epsilon > 0$ *there exists a partition* P *of* $[a, b]$ *such that* $U(f, P) - L(f, P) < \epsilon$.

7.2 Properties of the Riemann Integral

Theorem 98. *Let* f *be a monotonic function on* $[a, b]$. *Then* f *is integrable.*

Theorem 99. *Let* f *be a continuous function on* $[a, b]$. *Then* f *is integrable on* $[a, b]$.

Theorem 100. *Let* f *and* g *be integrable functions on* $[a, b]$ *and let* $k \in \mathbb{R}$. *Then*

(a) kf *is integrable and* $\int_a^b kf = k \int f$, *and*

(b) $f + g$ *is integrable and* $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

Theorem 101. *Suppose that* f *is integrable on both* $[a, c]$ *and* $[c, b]$. *Then* f *is integrable on* $[a, b]$. *Furthermore,* $\int_a^b f = \int_a^c f + \int_c^b f$.

Theorem 102. *Suppose that* f *is integrable on* $[a, b]$ *and* g *is continuous on* $[c, d]$, *where* $f([a, b]) \subseteq [c, d]$. *Then* $g \circ f$ *is integrable on* $[a, b]$.

Corollary 7. *Let* f *be integrable on* $[a, b]$. *The* $|f|$ *is integrable on* $[a, b]$ *and* $\Big| \displaystyle\int_a^b f \Big| \leq \displaystyle\int_a^b |f|$.

7.3 The Fundamental Theorem of Calculus

Theorem 103. *(The Fundamental Theorem of Calculus I) Let* f *be integrable on* $[a,b]$*. For each* $x \in [a,b]$ *let* $F(x) = \int_a^x f(t)\,dt$*. Then* F *is uniformly continuous on* $[a,b]$*. Furthermore, if* f *is continuous at* $c \in [a,b]$*, then* F *is differentiable at* c *and* $F'(c) = f(c)$*.*

Theorem 104. *(The Fundamental Theorem of Calculus II) If* f *is differentiable on* $[a,b]$ *and* f' *is integrable on* $[a,b]$*, then* $\int_a^b f' = f(b) - f(a)$*.*

Chapter 8

Infinite Series

8.1 Convergence of Infinite Series

Theorem 105. *Suppose that* $\sum a_n = s$ *and* $\sum b_n = t$*. Then* $\sum (a_n + b_n) = s + t$ *and* $\sum (ka_n) = ks$*, for every* $k \in \mathbb{R}$*.*

Theorem 106. *If* $\sum a_n$ *is a convergent series, then* $\lim a_n = 0$*.*

Theorem 107. *(Cauchy Criterion for Series) The infinite series* $\sum a_n$ *converges iff for each* $\epsilon > 0$ *there exists a number* N *such that if* $n \geq m > N$*, then* $|a_m + a_{m+1} + \cdots + a_n| < \epsilon$*.*

8.2 Convergence Tests

Theorem 108. *(Comparison Test) Let* $\sum a_n$ *and* $\sum b_n$ *be infinite series of nonnegative terms. That is,* $a_n \geq 0$ *and* $b_n \geq 0$ *for all* n*. Then*

1. *If* $\sum a_n$ *converges and* $0 \leq b_n \leq a_n$ *for all* n*, then* $\sum b_n$ *converges.*
2. *If* $\sum a_n = +\infty$ *and* $0 \leq a_n \leq b_n$ *for all* n*, then* $\sum b_n = +\infty$*.*

Theorem 109. *If a series converges absolutely, then it converges.*

Theorem 110. *(Ratio Test) Let* $\sum a_n$ *be a series of nonzero terms.*

1. *If* $\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1$*, then the series converges absolutely.*
2. *If* $\liminf \left| \frac{a_{n+1}}{a_n} \right| > 1$*, the the series diverges.*

3. *Otherwise,* $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$ *and the test gives no information about convergence or divergence.*

Theorem 111. *(Root Test) Given a series* $\sum a_n$*, let* $\alpha = \limsup |a_n|^{\frac{1}{n}}$*.*

1. *If* $\alpha < 1$*, then the series converges absolutely.*
2. *If* $\alpha > 1$*, then the series diverges.*
3. *Otherwise,* $\alpha = 1$ *and the test gives no information about convergence or divergence.*

Theorem 112. *(Integral Test) Let* f *be a continuous function defined on* $[0, \infty)$*, and suppose that* f *is positive and decreasing. That is, if* $x_1 < x_2$*, then* $f(x_1) \geq f(x_2) > 0$*. Then the series* $\sum(f(n))$ *converges iff* $\lim_{n\to\infty}\left(\int_1^n f(x)\,dx\right)$ *exists as a real number.*

Theorem 113. *(Alternating Series Test) If* (a_n) *is a decreasing sequence of positive numbers and* $\lim a_n = 0$*, then the series* $\sum(-1)^{n+1}a_n$ *converges.*

8.3 Power Series

Theorem 114. *Let* $\sum a_n x^n$ *be a power series and let* $\alpha = \limsup |a_n|^{\frac{1}{n}}$*. Define* R *by*

$$R = \begin{cases} \frac{1}{\alpha} & \text{if } 0 < \alpha < +\infty \\ +\infty & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha = +\infty \end{cases}.$$

Then the series converges absolutely whenever $|x| < R$ *and diverges whenever* $|x| > R$*. (When* $R = +\infty$ *we take this to mean that the series converges absolutely for all real* x*. When* $R = 0$ *then the series converges only at* $x = 0$*.)*

Theorem 115. *(Ratio Criterion) The radius of convergence* R *of a power series* $\sum a_n x^n$ *is equal to* $\lim \left|\frac{a_n}{a_{n+1}}\right|$*, provided that this limit exists.*

Chapter 9

Sequences and Series of Functions

9.1 Pointwise and Uniform Convergence

Theorem 116. *Let (f_n) be a sequence of functiond defined on a subset S of $\mathbb{R}$. There exists a function f such that (f_n) converges to f uniformly on S iff the following condition (called the Cauchy criterion) is satisfied:*

For every $\epsilon > 0$ there exists a number N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in S$ and all $m, n > N$.

Theorem 117. *(Weierstrass M-test) Suppose that (f_n) is a sequence of functions defined on S and (M_n) is a sequence of nonnegative numbers such that $|f_n(x)| \leq M_n$ for all $x \in S$ and all $n \in \mathbb{N}$. If $\sum M_n$ converges, then $\sum f_n$ converges uniformly on S.*

9.2 Applications Of Uniform Convergence

Theorem 118. *Let (f_n) be a sequence of continuous functions defined on a set S and suppose that (f_n) converges uniformly on S to a function $f : S \to \mathbb{R}$. Then f is continuous on S.*

Corollary 8. *Let $\sum\limits_{n=0}^{\infty} f_n$ be a series of functions defined on a set S. Suppose that each f_n is continuous on S and that the series converges uniformly to a function f on S. Then $f = \sum\limits_{n=0}^{\infty} f_n$ si continuous on S.*

Theorem 119. *Let (f_n) be a sequence of continuous functions defined on an interval $[a, b]$ and suppose that (f_n) converges uniformly on $[a, b]$ to a function*

f. Then $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

Corollary 9. *Let $\sum_{n=0}^{\infty} f_n$ be a series of functions defined on an interval $[a,b]$. Suppose that each f_n is continuous on $[a,b]$ and that the series converges uniformly to a function f on $[a,b]$. Then $\int_a^b f(x)\,dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)\,dx$.*

Theorem 120. *Suppose that (f_n) converges to f on an interval $[a,b]$. Suppose also that each f'_n exists and is continuous on $[a,b]$, and that the sequence (f'_n) converges uniformly on $[a,b]$. Then $\lim_{n\to\infty} f'_n(x) = f'(x)$ for each $x \in [a,b]$.*

Corollary 10. *Let $\sum_{n=0}^{\infty} f_n$ be a series of functions that converges to a function f on an interval $[a,b]$. Suppose that for each n, f'_n exists and is continuous on $[a,b]$ and that the series of derivatives $\sum_{n=0}^{\infty} f'_n$ is uniformly convergent on $[a,b]$. Then $f'(x) = \sum_{n=0}^{\infty} f'_n(x)$ for all $x \in [a,b]$.*

Theorem 121. *There exists a continuous function defined on $\mathbb{R}$ that is nowhere differentiable.*

9.3 Uniform Convergence of Power Series

Theorem 122. *Let $\sum a_n x^n$ be a power series with radius of convergence R, where $0 < R \leq +\infty$. If $0 < K < R$, then teh power series converges uniformly on $[-K,K]$.*

Theorem 123. *Suppose that a pwer series converges to a function f on $(-R,R)$, where $R > 0$. Then the series can be differentiated term by term, and the differentiated series converges on $(-R,R)$ to f'. That is, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, and both series have the same radius of convergence.*

Corollary 11. *Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-R,R)$, where $R > 0$. Then for each $k \in \mathbb{N}$, the kth derivative $f^{(k)}$ of f exists on $(-R,R)$ and*

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} \\ &= k! a_k + (k+1)! a_{k+1} + \frac{(k+2)!}{2!} a_{k+2} x^2 + \cdots . \end{aligned}$$

Furthermore, $f^{(k)}(0) = k!a_k$.

Corollary 12. *If* $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ *for all* x *in some interval* $(-R, R)$, *where* $R > 0$, *then* $a_n = b_n$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Theorem 124. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with a finite positive radius of convergence* R. *If the series converges at* $x = R$, *then it converges uniformly on teh interval* $[0, R]$. *Similarly, if the series converges at* $x = -R$, *then it converges uniformly on* $[-R, 0]$.

Corollary 13. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *have a finite positive radius of convergence* R. *If the series converges at* $x = R$, *then* f *is continuous at* $x = R$. *If the series converges at* $x = -R$, *then* f *is continuous at* $x = -R$.