§2.1 Basic Set Operations

The "idea" of a set is a collection of things with a common attribute.

Ex: Football team
flock of geese

The objects in a set are elements or members of the set. We denote sets by capital letters, usually.

Ex: $x \in A \Rightarrow x$ is an element of the set A
$5 \in \mathbb{R}$
$\pi \notin \mathbb{Z}$.

A set must be characterized by some defining property. There can't be a doubt as to whether an object belongs or not. i.e. the statement $a \in A$ must be true or false, not both.

Ex: Which of the following are sets?
a) all the current U.S. senators
b) all the tall people in Canada.
c) all the prime numbers between 8 and 10.

Examples of sets:
$A = \{1, 3, 5, 7\}$, $B = \{b\}$, $C = \{x \mid x \text{ is prime}\}$.

In listing the members of a set order or repitition is not important.

Def: Let A and B be sets. A is a subset of B, or A is contained in B, (denoted $A \subseteq B$), if every element in A is an element of B.
Other notations: $B \supseteq A$, $A \subset B$.

Def: Let A and B be sets. $A = B$ if $A \subseteq B$ and $B \subseteq A$. A is a proper subset of B if $A \subseteq B$ and $A \neq B$.

Familiar Sets:
$\mathbb{N}$ = natural #'s
$\mathbb{Z}$ = integers
$\mathbb{Q}$ = rationals
$\mathbb{R}$ = reals

Intervals:

$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ closed interval

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$ open interval

$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$ half-open/half-closed interval

Ex: $A = \{1,3\}$

$B = \{3,5\}$

$C = \{1,3,5\}$

$D = \{x \in \mathbb{R} \mid x^2 - 8x + 15 = 0\}$

$A \subseteq C$ $\quad$ $1 \notin D$

$A \nsubseteq B$ $\quad$ $C \nsubseteq B$

$5 \in B$ $\quad$ $B = D$

$\{5\} \subseteq B$ $\quad$ $B \neq C$.

Ex: Let $D = \{x \in \mathbb{Z} \mid x \text{ is prime and } 8 < x < 10\} = \emptyset$ = emptyset.

Theorem 4: Let A be a set. Then $\emptyset \subseteq A$.

Pf: To prove $\emptyset \subseteq A$, we must show if $x \in \emptyset$ then $x \in A$. Since $\emptyset$ has no members, the antecedent $x \in \emptyset$ is false for all x. Therefore the implication is always true.

Def: Let A and B be sets. The <u>union</u> of A and B, denoted $A \cup B$, is $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

The <u>intersection</u> of A and B is

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

The <u>complement</u> of B in A is

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

If $A \cap B = \emptyset$, then A and B are <u>disjoint</u>.

Ex: If $A=[1,3]$, $B=(1,5)$, what is $A\cap B$, $A\cup B$, $A\setminus B$, $B\setminus A$?

a) $A\cap B=(1,3]$

b) $A\cup B=[1,5)$

c) $A\setminus B=\{1\}$

d) $B\setminus A=(3,5)$

If in context we have a universal set X, we use the notation B^c for $X\setminus B$. In most cases our universal set will be $\mathbb{R}$.

Ex: If $B=[0,5)$ what is B^c?

$$B^c=\mathbb{R}\setminus[0,5) = (-\infty,0)\cup[5,\infty).$$

Theorem 5: Let A and B be subsets of $\mathcal{U}$. Then

$$A\cap(\mathcal{U}\setminus B)=A\setminus B.$$

Pf: We must show $A\cap(\mathcal{U}\setminus B)\subseteq A\setminus B$ and $A\cap(\mathcal{U}\setminus B)\supseteq A\setminus B$, or equivalently,

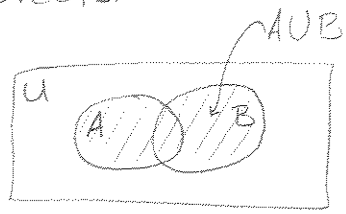
$$x\in A\cap(\mathcal{U}\setminus B) \text{ iff } x\in A\setminus B.$$

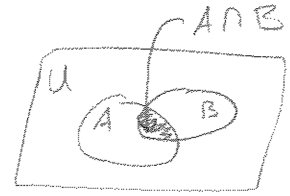
First we will show ($\subseteq$), so assume $x\in A\cap(\mathcal{U}\setminus B)$. Then $x\in A$ and $x\in \underline{\mathcal{U}\setminus B}$, by the definition of intersection. But $x\in\mathcal{U}\setminus B$ means $x\in\mathcal{U}$ and $\underline{x\notin B}$. Since $x\in A$ and $x\notin B$, we have $x\in\underline{A\setminus B}$, as required. Thus $A\cap(\mathcal{U}\setminus B)\subseteq A\setminus B$.

Conversely we must show $\underline{A\setminus B}\subseteq\underline{A\cap(\mathcal{U}\setminus B)}$. If $\underline{x\in A\setminus B}$, then $x\in A$ and $x\notin B$. Since $A\subseteq\mathcal{U}$, we have $x\in\underline{\mathcal{U}}$. Thus $x\in\mathcal{U}$ and $x\notin B$, so $\underline{x\in\mathcal{U}\setminus B}$. But then $\underline{x\in A}$ and $x\in\mathcal{U}\setminus B$, so $x\in A\cap(\mathcal{U}\setminus B)$.

Hence $A\setminus B\subseteq A\cap(\mathcal{U}\setminus B)$. $\boxed{}$

Venn Diagrams: Useful for visualizing, but they are not proofs.





Theorem 6: Let A, B and C be subsets of a universal set U. Then the following are true.

a) $A \cup (U \setminus A) = U$

b) $A \cap (U \setminus A) = \varnothing$

c) $U \setminus (U \setminus A) = A$

d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

f) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

g) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Pf: (d) Fill in the blanks.

(f) Fill in the blanks.

Theorem 7: If A and B are subsets of a set U and A^c and B^c are their complements in U, then:

a) $(A \cup B)^c = A^c \cap B^c$

b) $(A \cap B)^c = A^c \cup B^c$.

Def: If for any $j \in J$, where J is a nonempty set, there corresponds a set A_j, then $\mathcal{A} = \{A_j \mid j \in J\}$ is an <u>indexed family</u> of sets with J as the index set.

The union is $\bigcup \{A_j \mid j \in J\} = \{x \mid x \in A_j \text{ for some } j \in J\}$

The intersection is $\bigcap \{A_j \mid j \in J\} = \{x \mid x \in A_j \text{ for all } j \in J\}$.

Other notations:

$\bigcup \{A_j \mid j \in J\}$

$\bigcup_{j \in J} A_j$

$\bigcup \mathcal{A}$

If $J = \{1, \dots, n\}$ we may write $\bigcup_{j=1}^{n} A_j$.

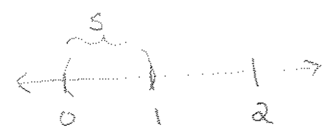
If $J = \mathbb{N}$, we write $\bigcup_{j=1}^{\infty} A_j$

Similar for intersections.

Def: If $\mathcal{A}$ is a collection of sets, $\mathcal{A}$ is called <u>pairwise disjoint</u> if $A \cap B = \emptyset$ $\forall A, B \in \mathcal{A}$ and $A \neq B$.

Ex: $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ where $A_n = (n, n+1)$. $\mathcal{A}$ is pairwise disjoint

Ex: If $\mathcal{A}$ is the collection of all intervals of the form $[s, 2]$ where $0 < s < 1$, find $\bigcap \mathcal{A}$ and $\bigcup \mathcal{A}$.



What do you think $\bigcap \mathcal{A}$ is? $[1, 2]$. Prove it!

Claim: $[1, 2] = \bigcap \mathcal{A}$.

Pf: $(\subseteq)$. Let $x \in [1, 2]$. Then $\forall\, s \in (0, 1)$, $x \in [s, 2]$, so $x \in \bigcap \mathcal{A}$

$(\supseteq)$. Let's prove if $y \notin [1, 2]$ then $y \notin \bigcap \mathcal{A}$. Clearly if $y > 2$, $y \notin \bigcap \mathcal{A}$, and if $y < 0$, $y \notin \bigcap \mathcal{A}$. Let $y \in (0, 1)$, and let $s = \frac{y+1}{2}$ = midpoint between y and 1. Then $y \notin [s, 2]$, so $y \notin \bigcap \mathcal{A}$. $\therefore \bigcap \mathcal{A} = [1, 2]$. ∎

What do you think $\bigcup \mathcal{A}$ is? $(0, 2]$. Prove it!

Claim: $(0, 2] = \bigcup \mathcal{A}$.

Pf: Let $x \in (0, 2]$. If $x \geq 1$, let $s = \frac{1}{2}$. Then $x \in [\frac{1}{2}, 2]$. If $0 < x$ let $s = \frac{x}{2}$. Then $x \in [\frac{x}{2}, 2]$, so $(0, 2] \subseteq \bigcup \mathcal{A}$.

For $(\supseteq)$, we will prove if $y \notin (0, 2]$, then $y \notin \bigcup \mathcal{A}$. Let $y \leq 0$. Th Clearly $y \notin [s, 2]$, for all $s \in (0, 1)$, so $y \notin \bigcup \mathcal{A}$. Similarly for $y >$ ∎