CH. 9: Sequences & Series of Functions

§9.1: Pointwise & Uniform Convergence

Def: Let (f_n) be a sequence of functions defined on a subset S of $\mathbb{R}$. Then (f_n) <u>converges pointwise</u> on S if for each $x \in S$, the sequence of numbers $(f_n(x))$ converges. If (f_n) converges pointwise on S, then we define $f: S \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for each $x \in S$, and we say that (f_n) converges to f pointwise on S.

$f_n: [0,1] \to [0,1]$

$f_n(x) = \begin{cases} 1-x^n & x \in \mathbb{Q} \\ x^n & x \notin \mathbb{Q} \end{cases}$

What properties are preserved when we take a limit, for example if f_n is continuous for all n, is f continuous? Or if f_n is differentiable (or integrable), is f?

Let's develop (explore) this question of continuity. If $S \subseteq \mathbb{R}$ is an interval and $x \in S$, then f is continuous at x iff

$$\lim_{t \to x} f(t) = f(x)$$

But if f_n is continuous at x $\forall n$, then

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left[\lim_{t\to x} f_n(t)\right].$$

Thus $f(t)$ will be continuous at x if

$$\lim_{t\to x}\left[\lim_{n\to\infty} f_n(t)\right] = \lim_{n\to\infty}\left[\lim_{t\to x} f_n(t)\right],$$

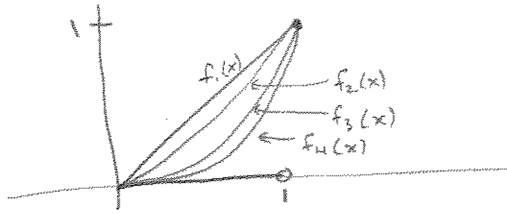
i.e. can we change the order of taking limits?

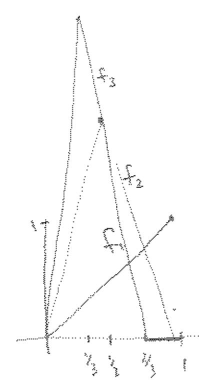
Since derivatives and integrals also are given by limits, the same idea holds.

Ex: For $x \in [0,1]$ and $n \in \mathbb{N}$, define $f_n(x) = x^n$. Then for $x \in [0,1)$ we have $\lim_{n\to\infty} f_n(x) = 0$ and $\lim_{n\to\infty} f_n(1) = 1$. Thus (f_n) is pointwise convergent on $S = [0,1]$, and the limit function f is given by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1. \end{cases}$$

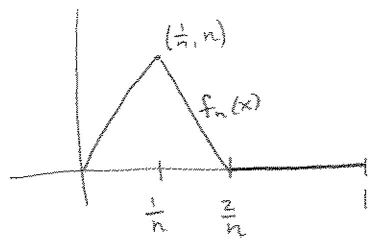
Notice that for each n, $f_n(x)$ is continuous and diff'ble on $[0,1]$, but f is neither continuous nor diff'ble, at $x = 1$.





Ex: For $x \in [0,1]$ and $n \geq 2$ define

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x \leq \frac{1}{n} \\ -n^2\left(x - \frac{2}{n}\right) & \text{if } \frac{1}{n} < x \leq \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} < x \leq 1. \end{cases}$$



Given any $x > 0$, let $M = \frac{2}{x}$. Then for $n > M$ we have $\frac{2}{n} < \frac{2}{M} = x$, so that $f_n(x) = 0$. Since $f_n(0) = 0$ $\forall n$, the limit function $f \equiv 0$. i.e. $f(x) = 0$ $\forall x \in [0,1]$. Each f_n is continuous, so it is integrable and $\int_0^1 f_n(x)\,dx = 1$ $\forall n \geq 2$

but $\lim_{n\to\infty} \left[\int_0^1 f_n(x)\,dx\right] = 1 \neq 0 = \int_0^1 \left[\lim f_n(x)\right] dx$

Def: Let (f_n) be a sequence of functions defined on a subset S of $\mathbb{R}$. Then (f_n) converges uniformly on S to a function f defined on S if

$$\forall \varepsilon > 0 \ \exists N \text{ s.t. } |f_n(x) - f(x)| < \varepsilon \quad \forall x \in S \text{ and all } n > N.$$

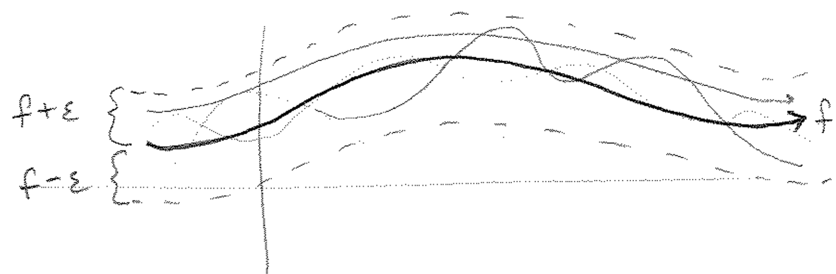
To say that a sequence (f_n) converges uniformly on S is to say that there exists a function f to which (f_n) converges uniformly on S.

Note the difference between pointwise & uniform convergence:

pointwise depends on ε and x.

uniform work for all $x \in S$.

Similar to the difference between continuous and uniformly continuous.



All f_n with $n > N$ lie in the ε-strip around f.

Ex: $f_n(x) = x^n$. If $\varepsilon = \frac{1}{2}$, then each f_n has values further then ε away from 0. Indeed, given any $n \in \mathbb{N}$, if $2^{-n} < x <$ then $f_n(x) > \frac{1}{2}$. Thus the convergence of f_n to 0 is not uniform on $[0,1]$.

Theorem ~~15~~ 16: Let (f_n) be a sequence of functions defined on a subset S of $\mathbb{R}$. Then $\exists$ a function f s.t. (f_n) converges to f uniformly on S iff the following condition (called the Cauchy criterion) is satisfied:

$$\forall \varepsilon > 0 \ \exists N \text{ s.t. } |f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in S \text{ and } \forall m,n > N.$$

Pf: Suppose the Cauchy criterion is satisfied. Since for each $x \in S$ $(f_n(x))$ is a Cauchy sequence, Theorem 60 implies that each sequence $(f_n(x))$ is convergent. Thus for each $x \in S$ we may define $f(x) = \lim_{n\to\infty} f_n(x)$, and the sequence of functions (f_n) converges pointwise on S to f. We claim this convergence is actually uniform.

Given any $\varepsilon > 0$, $\exists N$ s.t. $m, n > N \Rightarrow |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$ $\forall x \in S$

Fix n in this inequality and take the limit as $m \to \infty$. Since $f_m(x) \to f(x)$ as $m \to \infty$ we have

$$|f_n(x) - f(x)| \le \frac{\varepsilon}{2} < \varepsilon \quad \forall x \in S.$$

Thus (f_n) converges uniformly on S to f.

The converse implication is an exercise.

Note: Series of functions are handled in the same way analogous to series of constants and power series. If $(f_n)_{n=0}^{\infty}$ is a sequence of functions defined on a set S, the series $\sum_{n=0}^{\infty} f_n$ is said to converge pointwise (resp. uniformly) on S iff the sequence $(S_n)_{n=0}^{\infty}$ of partial sums, given by $S_n = \sum_{i=0}^{n} f_i(x)$, converges pointwise (resp. uniformly) on S.

Useful Test for establishing the uniform convergence of a series of functions

Theorem ~~116~~ 117: (Weierstrass M-test) Suppose that (f_n) is a sequence of functions defined on S and (M_n) is a sequence of nonnegative numbers s.t. $|f_n(x)| \le M_n$ $\forall x \in S$ and all $n \in \mathbb{N}$.

If $\sum M_n$ converges, then $\sum f_n$ converges uniformly on S.

Pf: We will show the partial sums, $S_n(x) = \sum_{k=0}^{n} f_k(x)$, satisfy the Cauchy criterion of Theorem 115.

Let $\varepsilon > 0$. Since $\sum M_n$ converges, there is some N s.t. if $n \geq m > N$, then

$$M_m + M_{m+1} + \dots + M_n < \varepsilon. \qquad (\text{Since } M_k > 0 \text{ we don't need } |$$

Thus if $n > m > N$ we have

$$\begin{aligned} |S_n(x) - S_m(x)| &= |f_{m+1}(x) + \dots + f_n(x)| \\ &\leq |f_{m+1}(x)| + \dots + |f_n(x)| \\ &\leq M_{m+1} + \dots + M_n < \varepsilon \qquad \forall x \in S. \end{aligned}$$

Thus by Theorem 115 (S_n) converges uniformly on $S \Rightarrow \sum f_n$ converges uniformly on S. $\blacksquare$

Ex: $\sum_{n=0}^{\infty} f_n$ where $f_n(x) = \frac{x^n}{n!}$ $\forall x \in \mathbb{R}$. In a previous example we showed this series is pointwise convergent on $\mathbb{R}$. We will show it is not uniformly convergent on $\mathbb{R}$, but given any $t \in \mathbb{R}$ it is uniformly convergent on $[-t,t]$.

To show $\sum f_n$ is not unif. conv. on $\mathbb{R}$, we will show the sequence of partial sums (S_n) does not satisfy the Cauchy criterion. (T. 115)

Let $\varepsilon = 1$. Given any $n \in \mathbb{N}$ let $x_0 = n$. It follows that

$$|S_n(x_0) - S_{n-1}(x_0)| = |f_n(x_0)| = f_n(n) = \frac{n^n}{n!} \geq 1 = \varepsilon.$$

$\therefore$ $\sum f_n$ is not unif. conv. on $\mathbb{R}$.

On the other hand if $t \in \mathbb{R}$, let $M_n = \frac{t^n}{n!}$. For any $x \in [-t,t]$ we have

$$|f_n(x)| = \left|\frac{x^n}{n!}\right| \leq \frac{t^n}{n!} = M_n.$$

Since $\sum M_n$ is convergent (by the ratio test) it follows from the Weierstrass M-test that $\sum \frac{x^n}{n!}$ is unif. conv. on $[-t, t]$.