§ 4.3: Monotone Sequences and Cauchy Sequences

Sometimes sequences converge but we are not able to find the value of the limit.

Def: A sequence (s_n) of real numbers is <u>nondecreasing</u> if $s_n \leq s_{n+1}$ $\forall n \in \mathbb{N}$, and is nonincreasing if $s_n \geq s_{n+1}$ $\forall n \in \mathbb{N}$. A sequence is <u>monotone</u> if it is either nonincreasing or nondecreasing.

Ex: $\left.\begin{array}{l} a_n = n \\ b_n = 2^n \\ c_n = 2 - \frac{1}{n} \end{array}\right\}$ all increasing sequences

$d_n = (1, 1, 2, 2, 3, 3, \ldots)$ is non-decreasing

$\left.\begin{array}{l} s_n = \frac{2}{n} \\ t_n = -3n \end{array}\right\}$ decreasing

$u_n = (1, 1, 1, 1, 1, \ldots)$ is both increasing and decreasing.

$\left.\begin{array}{l} x_n = \frac{(-1)^n}{n} \\ y_n = \cos\left(\frac{\pi n}{3}\right) \end{array}\right\}$ not monotonic

Theorem 58: (Monotone Convergence Theorem) A monotone sequence is convergent iff it is bounded.

Pf: Suppose that (s_n) is bounded and increasing. Let $S = \{s_n : n \in \mathbb{N}\}$, then S is bounded so by the completeness axiom, S has a least upper bound, so let $\sup S = s$. We claim $\lim s_n = s$. Given $\varepsilon > 0$, $s - \varepsilon < s$, so $\exists N$ s.t. $s_N > s - \varepsilon$. Also since (s_n) is increasing, $\forall n > N$, $s - \varepsilon < s_N < s_n < s$. $\therefore \lim s_n = s$.

In the case where (s_n) is decreasing, let $s = \inf S$, and proceed in a similar manner.

The converse implication was proven in Theorem 48. ∎

Ex: We want to show (s_n) converges where s_n is given by:

$$s_1 = 1, \quad s_{n+1} = \sqrt{1+s_n}.$$

Let's show this by showing (s_n) is increasing and bounded.

Claim: $s_n \leq s_{n+1}$ $\forall n$.

Proof by induction:

Let $n=1$. Then $s_1 = 1$, $s_2 = \sqrt{1+1} = \sqrt{2}$, $1 < \sqrt{2}$, so $n=1$ is true.

Assume $s_k \leq s_{k+1}$ for some $k \geq 1$. Then

$$s_{k+1} = \sqrt{s_k + 1} < \sqrt{1+s_{k+1}} = s_{k+2}.$$

$\therefore$ By induction, $s_n \leq s_{n+1}$ $\forall n \in \mathbb{N}$.

Claim: $s_n \leq 2$ $\forall n$.

Proof by induction.

Let $n=1$. Then $s_1 = 1 < 2$, so $n=1$ is true.

Assume $s_k \leq 2$ for some $k \geq 1$.

Then $s_{k+1} = \sqrt{1+s_k} < \sqrt{1+2} = \sqrt{3} < 2$.

$\therefore$ $s_n \leq 2$ $\forall n \in \mathbb{N}$.

Also (s_n) is bounded below by $s_1 = 1$, since it is increasing, so (s_n) is a bounded monotonic sequence $\Rightarrow$ it converges.

What does it converge to?

Since $\lim s_{n+1} = \lim s_n$, we see s must satisfy

$$s = \sqrt{s+1}$$

$$\Rightarrow \quad s^2 = s+1$$

$$s^2 - s - 1 = 0$$

$$s = \frac{1 \pm \sqrt{1+4}}{2}$$

$$s = \frac{1 \pm \sqrt{5}}{2}$$

But $s_n \geq 1 \Rightarrow s = \dfrac{1+\sqrt{5}}{2}$.

Theorem 59: a) If (s_n) is an unbounded increasing sequence then $\lim s_n = +\infty$

b) If (s_n) is an unbounded decreasing sequence, then $\lim s_n = -\infty$.

Pf: Let (s_n) be an increasing sequence and suppose that the set $S = \{s_n : n \in \mathbb{N}\}$ is unbounded. Since (s_n) is increasing, S is bounded below by s_1. Hence S must be unbounded above. Thus given any $M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ s.t. $s_N > M$. But then for any $n > N$ we have $s_n \geq s_N > M$, so $\lim s_n = +\infty$. The proof of (b) is similar. ☒

Cauchy Sequences

Def: A sequence (s_n) of real numbers is said to be a <u>Cauchy</u> sequence if

$$\forall \varepsilon > 0 \quad \exists N \text{ s.t. } m,n > N \Rightarrow |s_n - s_m| < \varepsilon.$$

Lemma 2: Every convergent sequence is a Cauchy sequence.

pf: Suppose $(s_n) \to s$. By the triangle inequality we have $|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m|$.

Thus given any $\varepsilon > 0$, choose N s.t. $k > N \Rightarrow |s_k - s| < \frac{\varepsilon}{2}$.

Then for $m, n > N$,

$$|s_n - s_m| \leq |s_n - s| + |s - s_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus (s_n) is a Cauchy sequence. ☒

Lemma 3: Every Cauchy sequence is bounded.

pf: Exercise.

Theorem 60: (Cauchy convergence Criterion) A sequence of real numbers is convergent iff it is a Cauchy sequence.

Pf: We have already shown $(\Rightarrow)$ Lemma 2. For the converse, suppose (s_n) is Cauchy and let $S = \{s_n : n \in \mathbb{N}\}$.

If S is finite, then $\varepsilon = \min\{|s_k - s_m| : s_k, s_m \in S\} > 0$. Since (s_n) is Cauchy $\exists N$ s.t. $m, n > N \Rightarrow |s_n - s_m| < \varepsilon$. Let n_0 be the smallest integer greater then N. Given any $m > N$, s_m and s_{n_0} are in S, so if $|s_m - s_{n_0}| < \varepsilon$, then $|s_m - s_{n_0}| = 0$ since ε = min distance between any s_k's in S. Thus $s_{n_0} = s_m$ $\forall m > N$. $\therefore \lim s_n = s_{n_0}$.

Now suppose S is infinite. From lemma 3 we know S is bounded. Thus from the Bolzano-Weierstrass Theorem, $\exists s \in \mathbb{R}$ s.t. s is an accumulation point of S. We claim $(s_n) \to s$. Given any $\varepsilon > 0$ $\exists N$ s.t. $|x_n - x_m| < \frac{\varepsilon}{2}$ $\forall n, m > N$. Since s is an accumulation pt of S, the Nbhd $N(s; \varepsilon/2) = (s - \frac{\varepsilon}{2}, s + \frac{\varepsilon}{2})$ contains infinitely many pts of S. Thus $\exists m > N$ s.t. $s_m \in N(s; \frac{\varepsilon}{2})$. Hence $\forall n > N$ we have

$$|s_n - s| = |s_n - s_m + s_m - s| \le |s_n - s_m| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore \lim s_n = s$.

Note: Thrm 60 depends on $\mathbb{R}$ being complete since it uses Bolzano-Weierstrass. In fact it can be shown an Archimedean ordered field is complete iff the Cauchy convergence criterion holds. The notion of Cauchy sequences can be defined whenever there is a notion of distance. In this more general setting, a Cauchy sequence may not necessarily converge, but it will be bounded.

Ex: Let $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$.

Show (s_n) diverges.

If $m > n$,

$$s_m - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{m}$$

$$> \underbrace{\frac{1}{m} + \frac{1}{m} + \dots + \frac{1}{m}}_{m-n \text{ terms}}$$

$$= \frac{m-n}{m}$$

$$= 1 - \frac{n}{m}.$$

In particular when $m = 2n$, $s_{2n} - s_n > \frac{1}{2}$.

So (s_n) is not Cauchy $\Rightarrow$ (s_n) diverges.