Chapter 7 Integration

§7.1: The Riemann Integral

Def: Let $[a,b]$ be an interval in $\mathbb{R}$. A partition P of $[a,b]$ is a finite set of points $\{x_0, x_1, \ldots, x_n\}$ in $[a,b]$ s.t.

$$a = x_0 < x_1 < \ldots < x_n = b.$$

If P and Q are two partitions of $[a,b]$ with $P \subseteq Q$, then Q is a refinement of P.

Def: Suppose that f is a bounded function defined on $[a,b]$ and that $P = \{x_0, \ldots, x_n\}$ is a partition of $[a,b]$. For each $i = 1, \ldots, n$ we let $M_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, and $m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. (We abbreviate these to M_i and m_i, respectively.) Letting $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \ldots, n$, we define the upper sum of f w.r.t. P to be

$$U(f,P) = \sum_{i=1}^{n} M_i(\Delta x_i),$$

and the lower sum of f w.r.t. P as

$$L(L,P) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Since we are assuming f is a bounded function on $[a,b]$ $\exists$ $m, M \in \mathbb{R}$ s.t. $m \le f(x) \le M$ $\forall x \in [a,b]$. Thus

$$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a).$$

$\therefore$ inf and sup of $L(f,P)$ and $U(f,P)$ exist.

Def: Let f be a bounded function defined on $[a,b]$. Then

$$U(f) = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

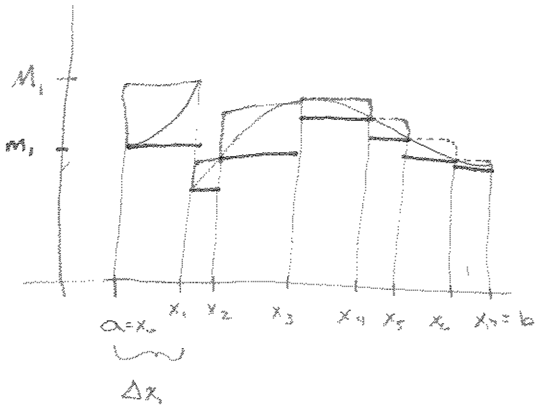
is the upper integral of f on $[a,b]$. Also

$$L(f) = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$$ is

the lower integral of f on $[a,b]$.

If $U(f) = L(f)$ we say f is Riemann integrable on $[a,b]$ and we

Ex: Pictorially



Since we are only going to discuss Riemann integrals, we simply refer to a function f as being <u>integrable</u> on $[a,b]$ and call $\int_a^b f$ the <u>integral</u> of f on $[a,b]$.

If f is nonnegative on $[a,b]$ we can interpret $\int_a^b f$ "intuitively" as the area under the graph. Not precise since we don't define "area".

Theorem 94: Let f be a bounded function on $[a,b]$. If P and Q are partitions of $[a,b]$ and Q is a refinement of P, then

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$$

Pf: $L(f,Q) \le U(f,Q)$ follows from the definition. To Prove $L(f,P) \le L(f,Q)$

Let $P = \{x_0, \dots, x_n\}$ and let P^* be the partition P with one more point, x^*, added. So $x_{k-1} < x^* < x_k$ for some $k = 1, \dots, n$.

Let $t_1 = \inf\{f(x) : x \in [x_{k-1}, x^*]\}$

$t_2 = \inf\{f(x) : x \in [x^*, x_k]\}$.

Then $t_1 \ge m_k$ and $t_2 \ge m_k$ where $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$.

The terms in $L(f,P^*)$ and $L(f,P)$ are all the same except those over the subinterval $[x_{k-1}, x_k]$. Thus we have

$$L(f,P^*) - L(f,P) = [t_1(x^* - x_{k-1}) + t_2(x_k - x^*)] - [m_k(x_k - x_{k-1})]$$

$$= (t_1 - m_k)(x^* - x_{i-1}) + (t_2 - m_k)(x_k - x_i^*) \ge 0.$$

So if the partition Q contains r more points then P, we apply the same argument r times to obtain $L(f,P) \le L(f,Q)$. The proof of $U(f, \dots$

Tues

Practice: Let f be a bounded function on $[a,b]$. If P and Q are partitions of $[a,b]$, prove that $L(f,P) \le U(f,Q)$.

Hint: Consider the partition $P \cup Q$ which is a refinement of both P + Q.

Pf: Since $P \cup Q$ is a refinement of both P and Q by the previous theorem we have

$$L(f,P) \le L(f, P\cup Q) \le U(f, P \cup Q) \le U(f,Q).$$

Theorem 95: Let f be a bounded function on $[a,b]$. Then $L(f) \le U(f)$.

Pf: If P and Q are partitions of $[a,b]$, then by the above exercise $L(f,P) \le U(f,Q)$. Thus $U(f,Q)$ is an upper bound for the set $S = \{L(f,P)\}$. So $U(f,Q)$ is at least as large as $\sup S = L(f)$, i.e. $L(f) \le U(f,Q)$ for each partition Q.

Then $L(f) \le \inf \{U(f,Q) : Q \text{ is a partition of } [a,b]\} = U(f)$.

Ex: We want to evaluate $\int_0^1 x^2\,dx$. For each $n \in \mathbb{N}$ let P_n be the partition

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}, \text{ so } \Delta x_i = \frac{1}{n} \quad \forall i = 1, \dots, n.$$

Since $f(x) = x^2$ is an increasing function on $[0,1]$, on any subinterval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ we have

$$M_i = \sup\left\{f(x) : x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]\right\} = f\left(\frac{i}{n}\right) = \left(\frac{i}{n}\right)^2 \quad \text{and}$$

$$m_i = \inf\left\{f(x) : x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]\right\} = f\left(\frac{i-1}{n}\right) = \left(\frac{i-1}{n}\right)^2.$$

Thus $$U(f,P_n) = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) = \frac{1}{n^3}\left(1^2 + 2^2 + \dots + n^2\right) = \frac{1}{n^3}\left[\frac{1}{6}(n)(n+1)(2n+1)\right]$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{2n}\right)$$

$$L(f,P_n) = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \left(\frac{1}{n}\right) = \frac{1}{n^3}\left(0^2 + 1^2 + \dots + (n-1)^2\right) = \frac{1}{n^3}\left[\frac{1}{6}(n-1)(n)(2n-1)\right]$$

$$= \frac{1}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{2n}\right)$$

$\lim U(f,P_n) = \frac{1}{3} = \lim L(f,P_n)$ so $U(f) \le \frac{1}{3}$ and $L(f) \ge \frac{1}{3}$ but

$L(f) \le U(f)$ by Theorem 95. $\therefore$ $L(f) = U(f) = \frac{1}{3}$, so

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

Ex: $g : [0,2] \to \mathbb{R}$ given by $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

Let $P = \{x_0, \dots, x_n\}$ be any partition of $[0,2]$. Since each subinterval $[x_{i-1}, x_i]$ contains rationals and irrationals, we have $M_i = 1$ and $m_i = 0$, for all $i = 1, \dots, n$. Thus

$$U(g,P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 2 \quad \text{and}$$

$$L(g,P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum 0 = 0.$$

It follows that $U(g) = 2$ and $L(g) = 0$. $\therefore$ g is not integrable on $[0,2]$.

Theorem 96: Let f be a bounded function on $[a,b]$. Then f is integrable iff ~~for~~ each $\varepsilon > 0$ $\exists$ a partition P of $[a,b]$ s.t.

$$U(f,P) - L(f,P) < \varepsilon.$$

Pf: Suppose f is integrable, so that $L(f) = U(f)$. Given $\varepsilon > 0$, $\exists$ a partition P_1 of $[a,b]$ s.t.

$$L(f, P_1) > L(f) - \frac{\varepsilon}{2}. \qquad (\text{Since } L(f) \text{ is a supremum}).$$

Similarly $\exists$ a partition P_2 of $[a,b]$ s.t.

$$U(f, P_2) < U(f) + \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$. Then by Theorem 94

$$U(f,P) - L(f,P) \le U(f, P_2) - L(f, P_1) < \left[U(f) + \frac{\varepsilon}{2}\right] - \left[L(f) - \frac{\varepsilon}{2}\right]$$

$$= U(f) - L(f) + \varepsilon = \varepsilon.$$

Conversely, given $\varepsilon > 0$ suppose $\exists$ a partition P of $[a,b]$ s.t. $U(f,P) < L(f,P) + \varepsilon$. Then we have

$$U(f) \le U(f,P) < L(f,P) + \varepsilon \le L(f) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we must have $U(f) \le L(f)$. By Theorem 95 this implies $L(f) = U(f) \Rightarrow f$ is integrable. $\blacksquare$