Chapter 3: The Real Numbers

§3.1: Natural Numbers and Induction

$\mathbb{N} = \{1, 2, 3, \ldots\}$

It is possible to "construct" the Natural numbers using Peano's axioms (see Taylor's notes for a taste) but we will not go through that development. We shall assume we can add and multiply natural numbers, and that we know what it means for one natural number to be less than another. There is one property we will accept: (proofs can be found in other sources)

<u>Axiom 1</u>: (The Well-ordering property of $\mathbb{N}$) If $S \subseteq \mathbb{N}$ and $S \neq \emptyset$, then $\exists m \in S$ s.t. $m \leq k$ $\forall k \in S$.

i.e. Every nonempty subset of $\mathbb{N}$ has a least element.

One important tool we have when dealing with $\mathbb{N}$ is induction. It allows us to conclude a given statement is true for all $n \in \mathbb{N}$ without checking them all one at a time.

Theorem 20: (Principle of Mathematical Induction) Let $P(n)$ be a statement that is either true or false for each $n \in \mathbb{N}$. Then $P(n)$ is true for all $n \in \mathbb{N}$ provided that

a) $P(1)$ is true, and

b) for each $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

Pf: (By contradiction, see tautology (g)). We will assume (a) and (b) hold, but $P(n)$ is false for some $n \in \mathbb{N}$. Let

$$S = \{n \in \mathbb{N} \mid P(n) \text{ is false}\}.$$

Then $S \neq \emptyset$, so S has a least element $m \in S$ s.t. $m \leq k$ $\forall k \in S$. Since $P(1)$ is true by hypothesis (a), $1 \notin S$, So it must be that $P(m-1)$ is true. Now hypothesis (b) tells us that if $k = m-1$, then $P(k+1) = P(m-1+1) = P(m)$ is true. This implies $m \notin S$, a contradiction. ⊠

We often refer to (a) of Theorem 20 as the basis for induction and (b) as the induction step. The assumption that $P(K)$ is true in (b) is the induction hypothesis. Both parts must be varified in a complete proof.

Theorem 21: $1+2+3+\dots+n = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$.

Pf: Let $P(n)$ be the statement $1+2+3+\dots+n = \frac{n(n+1)}{2}$.

Then $P(1)$ asserts that $1 = \frac{1}{2}(1)(1+1)$, which is true. We could continue and see that $P(2)$ is $1+2=3=\frac{2(3)}{2}$ which is also true. But showing $P(1)$ is true is our basis for induction.

Now assume $P(K)$ is true, that is we assume

$$1+2+\dots+K = \frac{K(K+1)}{2}.$$

Now we must show $P(K+1)$ is true, i.e. we want $1+2+\dots+K+(K+1) = \frac{(K+1)(K+2)}{2}$.

$$\begin{aligned}
\text{But } 1+2+\dots+K+(K+1) &= \left[1+2+\dots+K\right]+(K+1) \\
&= \frac{K(K+1)}{2}+(K+1) \\
&= \frac{K(K+1)+2(K+1)}{2} = \frac{(K+1)(K+2)}{2}
\end{aligned}$$

Therefore $P(K+1)$ is true whenever $P(K)$ is true. So by induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

- Theorem 22: $7^n - 4^n$ is a multiple of 3 $\forall n \in \mathbb{N}$.

Pf: (By induction)

Let $n=1$, then $7-4=3$ which is a multiple of 3.

Now Assume $7^k - 4^k$ is a multiple of 3, i.e. $7^k - 4^k = 3l$ for some $l \in \mathbb{N}$. Then $7^{k+1} - 4^{k+1} = 7\cdot 7^k - 7\cdot 4^k + 7\cdot 4^k - 4\cdot 4^k$

$$= 7(7^k - 4^k) + 4^k(7-4)$$

$$= 7(3l) + 4^k(3)$$

$$= 3(7l + 4^k).$$

Since l and k are natural numbers, $7l + 4^k \in \mathbb{N}$. Thus $7^{k+1} - 4^{k+1}$ is a multiple of 3. So by induction, $7^n - 4^n$ is a multiple of 3 $\forall n \in \mathbb{N}$. ☺

must get here

Theorem 23: (The Binomial Formula) Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $0! = 1$.

If $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Pf: (By induction) See Taylor's notes. Quite magical.

Theorem 24: Let $m \in \mathbb{N}$ and let $P(n)$ be a statement that is either true or false for each $n \geq m$. Then $P(n)$ is true for all $n \geq m$ provided that

(a) $P(m)$ is true, and

(b) for each $k \geq m$, if $P(k)$ is true, then $P(k+1)$ is true.

Pf: $\forall r \in \mathbb{N}$ let $Q(r)$ be the statement "$P(r+m-1)$ is true." From (a) we know $Q(1)$ is true. Now let $j \in \mathbb{N}$ and suppose $Q(j)$ is true, i.e. $P(j+m-1)$ is true. Since $j \in \mathbb{N}$, $j+m-1 = m+(j-1) \geq m$, so by (b) $P(j+m)$ must be true. Thus $Q(j+1)$ holds, and the induction step is verified. Thus $Q(r)$ hold $\forall r \in \mathbb{N}$. ☺

Recursive Relations:

Uses inductive reasoning to define elements in a set (sequence

Ex: $x_1 = 5 \quad x_{n+1} = x_n + 2$

$5, 7, 9, 11, \ldots$

The expression for x_{n+1} is usually some kind of function which refers to the previous (x_n) element. You can't get the 15^{th} term unless you know the 14^{th}.

Ex: Let $x_1 = 1$ and $x_{n+1} = \sqrt{x_n + 1}$. Show $x_m < x_{m+1} < 2$ $\forall m \in \mathbb{N}$.

Pf: (By induction) Let $n=1$. Then $x_1 = 1$, $x_2 = \sqrt{2}$

$1 < \sqrt{2} < 2$ is true, so we have verified our basis for induction.

Assume $x_k < x_{k+1} < 2$. We need to show $x_{k+1} < x_{k+2} < 2$.

But $x_{k+1} = \sqrt{x_k + 1}$

$x_{k+2} = \sqrt{x_{k+1} + 1}$

So simply take our assumed inequality, add 1 to each part and take a square root.

$$x_k < x_{k+1} < 2$$

$$\sqrt{x_k + 1} < \sqrt{x_{k+1} + 1} < \sqrt{3}$$

$$x_{k+1} < x_{k+2} < \sqrt{3} < 2$$

so we have verified our induction step.

$\therefore x_n < x_{n+1} < 2 \quad \forall n \in \mathbb{N}$. $\square$