Math 3210-3
HW 9
Solutions

NOTE: You are only required to turn in problems 1, 2, 5, and 6.

The Completeness Axiom

1. ♣ For each of the following subsets of $\mathbb{R}$, describe the set of all upper bounds for the set. If the set is not bounded above you can write "NOT BOUNDED ABOVE." Also give the supremum and maximum of the set, if they exist.

   (a) the set of odd integers; **Not Bounded Above.**

   (b) $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$; **Bounded by** $\{x \in \mathbb{R} : x \geq 1\}$**.** $\sup S = 1$**, and there is no maximum.**

   (c) $\{r \in \mathbb{Q} : r^3 < 8\}$; **Bounded by** $\{x \in \mathbb{R} : 2 \leq x\}$**.** $\sup S = 2$**,** $\max S =$ **does not exist.**

   (d) $\{\sin x : x \in \mathbb{R}\}$. **Bounded by** $\{x \in \mathbb{R} : x \geq 1\}$**.** $\sup S = 1$**,** $\max S = 1$**.**

2. Show that the set $A = \{x : x^2 < 1 - x\}$ is bounded above, and then find its least upper bound. (Make sure you prove that it is the least upper bound.)

   $A = \{x : x^2 < 1 - x\} = \{x : x^2 + x - 1 < 0\} = \left(\dfrac{-1-\sqrt{5}}{2}, \dfrac{-1+\sqrt{5}}{2} \right)$, so clearly A is bounded above by any $y \in \mathbb{R}$ with $y \geq \dfrac{-1+\sqrt{5}}{2}$. The least upper bound is $\dfrac{-1+\sqrt{5}}{2}$.

3. Let S be a nonempty bounded subset of R and let $k \in \mathbb{R}$. Define $kS = \{ks : s \in S\}$. Prove the following:

   (a) If $k \geq 0$, then $\sup(kS) = k \cdot \sup S$ and $\inf(kS) = k \cdot \inf S$.

   Proof: Let $x = \sup S$ and $y = \inf S$. Since $x \geq s$ for all $s \in S$, $y \leq s$ for all $s \in S$, and $k \geq 0$, we have $kx \geq ks$ and $yk \leq ks$ for all $s \in S$ by O4. So kx is an upper bound for kS and yk is a lower bound for kS. To finish the proof we need to show that if $z, w \in \mathbb{R}$ with $z < kx$ and $w > ky$, then there exists $r', t' \in kS$ such that $r' > z$ and $t' < w$. So let $z, w \in \mathbb{R}$ with $z < kx$ and $w > ky$. Then $\frac{z}{k} < x$ and $\frac{w}{k} > y$ which implies that $\exists r, t \in S$ with $r > \frac{z}{k}$ and $t < \frac{w}{k}$, which implies $rk > z$ and $tk < w$. Since $rk, tk \in kS$, we have $kx = \sup(kS)$ and $ky = \inf(kS)$.

   □

   (b) If $k < 0$, then $\sup(kS) = k \cdot \inf S$ and $\inf(kS) = k \cdot \sup S$.

   Proof: This proof is very similar to the previous one. We just need to recall from Theorem 25, that in this case, $xk < ks$ and $yk > ks$ for all $s \in S$ since $k < 0$. Thus xk is a lower bound for kS, and yk is an upper bound for kS. Let $z, w \in R$ with $z > kx$ and $w < ky$. Then $\frac{z}{k} < x$ and $\frac{w}{k} > y$ which implies that there exist $r, t \in S$ with $r > \frac{z}{k}$ and $t < \frac{w}{k}$. Hence $rk < z$ and $tk > w$, and $rk, tk \in kS$, so $\sup(kS) = k \cdot \inf S$ and $\inf(kS) = k \cdot \sup S$.

   □

4. Let A be a non-empty subset of $\mathbb{R}$ and x an element of $\mathbb{R}$. Then prove the following:

   (a) $\sup A \leq x$ iff $a \leq x$ for every $a \in A$.

   Proof: Suppose $\sup A \leq x$. Then for all $a \in A$, $a \leq \sup A \leq x$, so $a \leq x$. On the other hand, suppose that $a \leq x$ for every $a \in A$. Then x is a bound for A, but by definition, $\sup A \leq x$.

   □

   (b) $x < \sup A$ iff $x < a$ for some $a \in A$.

   Proof: Suppose $x < \sup A$. Then by definition, there exists $a \in A$ such that $x < a$. On the other hand, suppose $x < a$ for some $a \in A$. Then $x < a \leq \sup A$, so $x < \sup A$.

□

5. Let A and B be non-empty subsets of $\mathbb{R}$. Then prove the following:

   (a) ♣ $\sup(A - B) = \sup(A) - \inf(B)$.

   Proof: By Theorem 32, we know $\sup(A + (-B)) = \sup A + \sup(-B) = \sup A - \inf B$.

   □

   (b) If $A \subseteq B$, then $\sup A \leq \sup B$ and $\inf B \leq \inf A$.

   Proof: Suppose $A \subseteq B$ and $\sup A > \sup B$. Then by definition, there exists $a \in A$ such that $\sup B < a$. But $a \in B$ since $A \subseteq B$, so $\sup B < a \leq \sup B$, a contradiction. Therefore $\sup A \leq \sup B$.

   Now suppose $A \subseteq B$ and $\inf B > \inf A$. Then once again $\exists a \in A$ such that $a < \inf B$, but $a \in B$, so $\inf B \leq a < \inf B$, a contradiction. Therefore $\inf B \leq \inf A$.

   □

6. ♣ Prove: If $y > 0$, then there exists a unique $n \in \mathbb{N}$ such that $n - 1 \leq y < n$.

   Proof: Let $S = \{m \in \mathbb{N} : m > y\}$. Then by Axiom 1, there is a unique $n \in S$ such that $n \leq k$ for all $k \in S$. So $n > y$ and $n - 1 \leq y$ since $n - 1 \notin S$. Therefore n is our unique value for which $n - 1 \leq y < n$.

   □

7. Let $x \in \mathbb{Q}$, $x \neq 0$, and let y be irrational. Prove that xy is irrational.

   Proof: Suppose $x \in \mathbb{Q}, y \notin \mathbb{Q}$, and $xy \in \mathbb{Q}$. Then we can write $x = \frac{r}{s}$ and $xy = \frac{p}{q}$ for $r, s, p, q \in \mathbb{Z}$ and r and s have no common divisors, and p and q also have no common divisors. Then we have the following:

$$
\begin{aligned}
xy &= \frac{p}{q} \qquad \text{which implies that} \\
y &= \frac{p}{qx} \\
&= \frac{p}{q\frac{r}{s}} \\
&= \frac{ps}{qr}
\end{aligned}
$$

   But $\frac{ps}{qr} \in \mathbb{Q}$, a contradiction. Therefore xy is irrational.

   □