Math 3210-3
HW 8
Solutions

Ordered Fields

1. ♣ Let $x, y \in \mathbb{R}$. Prove $xy = 0$ iff $x = 0$ or $y = 0$.

   Proof: First assume $x = 0$. Then by Theorem 25(b), $xy = 0$. Similarly, if $y = 0$, $xy = 0$. For the reverse implication, assume $xy = 0$, and suppose $x \neq 0$. Then by M5 there exists $x^{-1} \neq 0$, so we have the following:

$$\begin{aligned} xy &= 0 \\ x^{-1}xy &= x^{-1}0 \\ y &= 0 \end{aligned}$$

□

2. ♣ Let $x, y, z \in \mathbb{R}$. Prove that if $x < y$ and $z < 0$, then $xz > yz$.

   Proof: If $z < 0$, then $-z > 0$, so we have:

$$\begin{aligned} x(-z) &< y(-z) && \text{by O4} \\ x(-1)z &< y(-1)z && \text{by Theorem 25} \\ (-1)xz &< (-1)yz && \text{by M2} \\ -(xz) &< -(yz) && \text{by Theorem 25} \\ xz &> yz && \text{by Theorem 25} \end{aligned}$$

□

3. Let $x, y \in \mathbb{R}$. Prove $|x \cdot y| = |x| \cdot |y|$.

   Proof: Case 1: Suppose $x, y \geq 0$. Then $xy \geq 0$, so we have $|x| = x, |y| = y$, and $|xy| = xy$, so $|x||y| = xy = |xy|$.

   Case 2: Suppose $x < 0$ and $y \geq 0$. Then $xy \leq 0$ and we have $|x| = -x, |y| = y$, and $|xy| = -(xy)$. Thus we have $|x||y| = (-x)y = -(xy) = |xy|$.

   Case 3: Suppose $x, y < 0$. Then $xy > 0$, so we have $|x| = -x, |y| = -y$, and $|xy| = xy$. Thus we have $|x||y| = (-x)(-y) = xy = |xy|$.

   Therefore for any $x, y \in \mathbb{R}$, we have $|x||y| = |xy|$.

□

4. (a) Prove that $|a + b + c| \leq |a| + |b| + |c|$ for all $a, b, c \in \mathbb{R}$.

   Proof: Notice that $|a + b + c| = |(a + b) + c| \leq |a + b| + |c| \leq (|a| + |b|) + |c| = |a| + |b| + |c|$.

□

   (b) Use induction to prove $|a_1 + a_2 + a_3 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$ for n numbers $a_1, a_2, \ldots, a_n$.

   Proof: By Theorem 27, we know this is true for $n = 1$ and $n = 2$. Also by part (a) we know $n = 3$ is true. Assume that $|a_1 + \cdots + a_k| \leq |a_1| + \cdots + |a_k|$ for some $k \geq 3$. Then $|a_1 + \cdots + a_k + a_{k+1}| = |(a_1 + a_2 + \cdots + a_k) + a_{k+1}| \leq |a_1 + \cdots + a_k| + |a_{k+1}| \leq |a_1| + |a_2| + \cdots + |a_k| + |a_{k+1}|$, by the triangle inquality. Therefore $|a_1 + a_2 + a_3 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$.

□

5. ♣ Prove that in any ordered field F, $a^2 + 1 > 0$ for all $a \in F$. Conclude from this that if the equation $x^2 + 1 = 0$ has a solution in a field, then that field cannot be ordered. (Thus, it is not possible to define an order relation on the set of all complex numbers that will make it an ordered field.)

   Proof: Let F be an ordered field. First notice that if $a < 0$, then by problem 2, $a \cdot a > a \cdot 0 = 0$. Also, if $a > 0$, then by O4, $a \cdot a > 0 \cdot a = 0$. In either case, $a^2 > 0$, so $a^2 + 1 > 0$. Finally, if $a = 0$, then $a^2 = 0$, so $a^2 + 1 = 1 > 0$.

   Now assume we have an ordered field F in which there is some $x \in F$ such that $x^2 + 1 = 0$. Then $x^2 + 1 = 0$ and $x^2 + 1 > 0$, which implies that $0 > 0$, a contradiction. So any field with a solution to $x^2 + 1 = 0$ can not be an ordered field.

   □