# Math 3210-3
## HW 7
Solutions

## The Natural Numbers and Induction

1. ♣ Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

   *Proof:* Let $P(n)$ be the statement $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$. First we need to show that $P(1)$ is true. But $P(1)$ asserts that $1^2 = 1 = \frac{(1)(1+1)(2+1)}{6} = 1$, so $P(1)$ is true. Now assume that $P(k)$ is true, i.e., $1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$. We must now show that $P(k+1)$ is true, i.e., $1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$ is true. But we have

$$\begin{aligned}
[1^2 + 2^2 + \cdots + k^2] + (k+1)^2 &= \frac{(k)(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

   Which is what we needed to show. Therefore $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

   □

2. ♣ Prove that $5^{2n} - 1$ is a multiple of 8 for all $n \in \mathbb{N}$.

   *Proof:* Let $n = 1$. Then $5^2 - 1 = 24 = 8 \cdot 3$ is a multiple of 8, so $n = 1$ is true. Assume $5^{2k} - 1$ is a multiple of 8 for some $k \in \mathbb{N}, k > 1$. In other words, $5^{2k} - 1 = 8l$ for some $l \in \mathbb{Z}$. We must now show $5^{2(k+1)} - 1$ is a multiple of 8. But $5^{2(k+1)} - 1 = 5^{2k+2} - 1 = 5^2 \cdot 5^{2k} - 1 = 5^2 \cdot 5^{2k} - 5^2 + 5^2 - 1 = 5^2(5^{2k} - 1) + 24 = 5^2(8l) + 8 \cdot 3 = 8(5^2 l + 3)$. So $5^{2(k+1)} - 1$ is a multiple of 8. Therefore $5^{2n} - 1$ is a multiple of 8 for all $n \in \mathbb{N}$.

   □

3. ♣ Let a sequence $\{x_n\}$ of numbers be defined recursively by

$$x_1 = 0 \text{ and } x_{n+1} = \frac{x_n + 1}{2}.$$

   Prove by induction that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. Would this conclusion change if we set $x_1 = 2$?

   *Proof:* Let $n = 1$. Then we see that $x_1 = 0$ and $x_2 = \frac{1}{2}$, and clearly $0 \leq \frac{1}{2}$, so $n = 1$ is true. Suppose that for $k \geq 1$, $x_k \leq x_{k+1}$. Then we need to show that $x_{k+1} \leq x_{k+2}$. So we have $x_k \leq x_{k+1}$, so add one to each side, and divide by two to get $x_{k+1} = \frac{x_k+1}{2} \leq \frac{x_{k+1}+1}{2} = x_{k+2}$, as we wanted. Therefore, $x_n \leq x_{n+1} \forall n \in \mathbb{N}$.

   Let $x_1 = 2$. Then $x_2 = \frac{3}{2}$, and $x_2 < x_1$, so the conclusion does not hold. Our new conclusion for this case would be $x_n \geq x_{n+1} \forall n \in \mathbb{N}$.

   □

4. For each $n \in \mathbb{N}$, let $P_n$ denote the assertion "$n^2 + 5n + 1$ is an even integer."

   (a) Prove that $P_{n+1}$ is true whenever $P_n$ is true.

   *Proof:* Assume $P_n$ is true, then $n^2 + 5n + 1 = 2k$ for some $k \in \mathbb{Z}$. Then $(n+1)^2 + 5(n+1) + 1 = n^2 + 2n + 1 + 5n + 5 + 1 = (n^2 + 5n + 1) + (2n + 6) = 2k + 2(n+3) = 2(k + n + 3)$, which is an even number, so $P_{n+1}$ is also true.

□

(b) For which $n$ is $P_n$ actually true? What is the moral of this exercise?

*Proof:* I claim that $n^2 + 5n + 1$ is odd for all $n \in \mathbb{N}$. If $n = 1$, then we have $1 + 5 + 1 = 7$, so $n = 1$ is true. Now suppose $k^2 + 5k + 1 = 2m - 1$ for some $m \in \mathbb{Z}$. Then $(k+1)^2 + 5(k+1) + 1 = (2m-1) + 2(n+3) = 2m + 2n + 5 = 2(m + k + 3) - 1$ which is odd. So $P_n$ is true for all $n \in \mathbb{N}$. The moral of this exercise is to show that it is necessary to verify $P_n$ for the base case.

□

5. ♣ Use Theorem 24 to prove that $n^2 < 2^n$ for all $n \geq 5$.

*Proof:* By Theorem 24, we need to show $n = 5$ is true, and when $n = k$ is true, then $n = k + 1$ is also true. Let $n = 5$. Then $5^2 = 25 < 32 = 2^5$, so $n = 5$ is true. Assume $k^2 < 2^k$ for some $k \geq 5$. Then $(k+1)^2 = k^2 + 2k + 1 < 2^k + 2k + 1$. If we can show that $2k + 1 < 2^k$, then we will have $(k+1)^2 < 2^k + 2^k = 2^{k+1}$. Then we can conclude that $n^2 < 2^n$ for all $n \in \mathbb{N}$. We will have this conclusion when we prove the next lemma.

□

**Lemma 1**
$2n + 1 < 2^n$ for all $n \geq 5$.

*Proof:* Let $n = 5$. Then $2(5)+1 = 11 < 32 = 2^5$, so $n = 5$ is true. Assume $2k+1 < 2^k$ for some $k \geq 5$. Then $2(k+1)+1 = 2k+3 = (2k+1)+2 < 2^k+2$, and $2 < 2^k$ for $k \geq 5$, so $2(k+1)+1 < 2^k+2^k = 2^{k+1}$. Therefore $2k + 1 < 2^n$ for all $n \geq 5$.

□